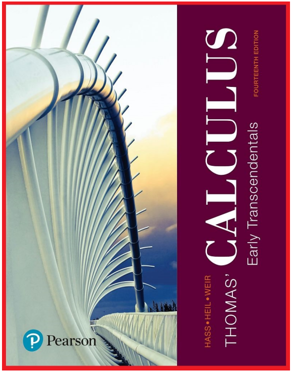
Calculus 1

Chapter 2. Limits and Continuity

2.6. Limits Involving Infinity; Asymptotes of Graphs—Examples and Proofs



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Suppose $x > M = 1/\varepsilon$ (notice then that x is positive). This implies $0 < 1/x < 1/M = \varepsilon$, or $0 < \frac{1}{x} = \left|\frac{1}{x} - 0\right| = |f(x) - L| < \varepsilon$.

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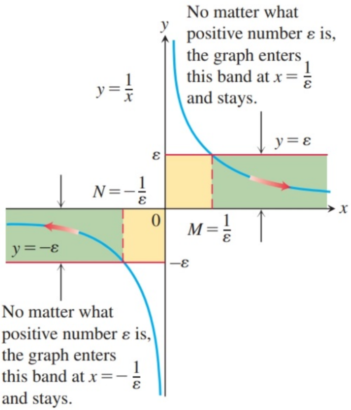

Figure 2.50

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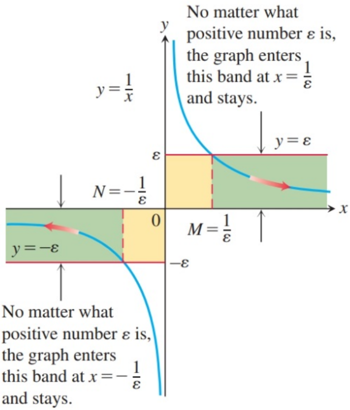

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□

Exercise 2.6.14

Exercise 2.6.14. For the rational function $f(x) = \dfrac{2x^3+7}{x^3-x^2+x+7}$, find the limit as (a) $x \to \infty$, and (b) $x \to -\infty$. Justify your computations with Theorem 2.12.

Solution. We can evaluate both limits by the same process. We have

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3+7}{x^3-x^2+x+7} \text{ by the definition of } f$$

$$= \lim_{x\to\pm\infty} \frac{2x^3+7}{x^3-x^2+x+7}\left(\frac{1/x^3}{1/x^3}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}$$

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$$= \lim_{x\to\pm\infty} \frac{(2x^3+7)/x^3}{(x^3-x^2+x+7)/x^3}$$

$$= \lim_{x\to\pm\infty} \frac{2x^3/x^3+7/x^3}{x^3/x^3-x^2/x^3+x/x^3+7/x^3}$$

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$$= \lim_{x\to\pm\infty} \frac{(2x^3+7)/x^3}{(x^3-x^2+x+7)/x^3}$$

$$= \lim_{x\to\pm\infty} \frac{2x^3/x^3+7/x^3}{x^3/x^3-x^2/x^3+x/x^3+7/x^3}$$

Exercise 2.6.14 (continued 1)

Solution (continued).

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3/x^3 + 7/x^3}{x^3/x^3 - x^2/x^3 + x/x^3 + 7/x^3}$$

$= \lim_{x\to\pm\infty} \dfrac{2 + 7/x^3}{1 - 1/x + 1/x^2 + 7/x^3}$ since $x \to \pm\infty$ then we can assume that $x \neq 0$

$= \dfrac{\lim_{x\to\pm\infty}(2 + 7/x^3)}{\lim_{x\to\pm\infty}(1 - 1/x + 1/x^2 + 7/x^3)}$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

$= \dfrac{\lim_{x\to\pm\infty} 2 + \lim_{x\to\pm\infty} 7/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + \lim_{x\to\pm\infty} 7/x^3)}$ by the Sum and Difference Rules (Theorem 2.12(1 and 2))

$= \dfrac{\lim_{x\to\pm\infty} 2 + 7\lim_{x\to\pm\infty} 1/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + 7\lim_{x\to\pm\infty} 1/x^3)}$

Exercise 2.6.14 (continued 1)

Solution (continued).

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x^3/x^3 + 7/x^3}{x^3/x^3 - x^2/x^3 + x/x^3 + 7/x^3}$$

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Exercise 2.6.14 (continued 2)

Solution (continued).

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\lim_{x\to\pm\infty} 1/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + 7\lim_{x\to\pm\infty} 1/x^3}$$

by the Constant Multiple Rule (Theorem 2.12(4))

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\left(\lim_{x\to\pm\infty} 1/x\right)^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \left(\lim_{x\to\pm\infty} 1/x\right)^2 + 7\left(\lim_{x\to\infty} 1/x\right)^3}$$

by the Power Rule (Theorem 2.12(6))

$$= \frac{(2) + 7(0)^3}{(1) - (0) + (0)^2 + 7(0)^3}$$ by Example 2.6.1

$$= \frac{2}{1} = \boxed{2}. \qquad \square$$

Exercise 2.6.14 (continued 2)

Solution (continued).

$$= \frac{\lim_{x\to\pm\infty} 2 + 7\lim_{x\to\pm\infty} 1/x^3}{\lim_{x\to\pm\infty} 1 - \lim_{x\to\pm\infty} 1/x + \lim_{x\to\pm\infty} 1/x^2 + 7\lim_{x\to\pm\infty} 1/x^3}$$

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Exercise 2.6.36

Exercise 2.6.36. Evaluate $\lim_{x \to -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$ by dividing the numerator and denominator by the (effective) highest power of x in the denominator. Justify your computations with Theorem 2.12.

Solution. We have a square root of x^6 in the denominator, so the "effective" highest power of x in the denominator is 3 (think what happens when x is really large: $x^6 + 9$ is about the same size as x^6 and $\sqrt{x^6 + 9}$ is about the same size as x^3).

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$$\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} \left(\frac{1/x^3}{1/x^3}\right) \text{ dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3}$$

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$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3}$$

Exercise 2.6.36 (continued 1)

Solution (continued).

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3} = \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/(-\sqrt{x^6})}$$

since $\sqrt{x^6} = |x^3| = -x^3$ for x negative

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{-\sqrt{(x^6+9)/x^6}} = \lim_{x\to-\infty} \frac{4/x^3 - 3x^3/x^3}{-\sqrt{x^6/x^6 + 9/x^6}}$$

$$= \lim_{x\to-\infty} \frac{4/x^3 - 3}{-\sqrt{1+9/x^6}}$$ since $x \to -\infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to-\infty}(4/x^3 - 3)}{\lim_{x\to-\infty}(-\sqrt{1+9/x^6})}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

Exercise 2.6.36 (continued 1)

Solution (continued).

$$= \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/x^3} = \lim_{x\to-\infty} \frac{(4-3x^3)/x^3}{(\sqrt{x^6+9})/(-\sqrt{x^6})}$$

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Exercise 2.6.36 (continued 2)

Solution (continued).

$$= \frac{\lim_{x\to-\infty}(4/x^3) - \lim_{x\to-\infty} 3}{-\lim_{x\to-\infty}(\sqrt{1+9/x^6})}$$ by the Difference Rule and the Constant Multiple Rule, Theorem 2.12(2 and 4)

$$= \frac{\lim_{x\to-\infty}(4/x^3) - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1+9/x^6)}}$$ by the Root Rule, Theorem 2.12(7)

$$= \frac{4\lim_{x\to-\infty}(1/x^3) - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\lim_{x\to-\infty}(1/x^6)}}$$ by the Sum Rule and Constant Multiple Rule, Theorem 2.12(1 and 4)

$$= \frac{4\left(\lim_{x\to-\infty} 1/x\right)^3 - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\left(\lim_{x\to-\infty} 1/x\right)^6}}$$ by the Power Rule, Theorem 2.12(6)

Exercise 2.6.36 (continued 2)

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Exercise 2.6.36 (continued 3)

Exercise 2.6.36. Evaluate $\lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$ by dividing the numerator and denominator by the (effective) highest power of x in the denominator. Justify your computations with Theorem 2.12.

Solution (continued).

$$\begin{aligned} \lim_{x\to-\infty} \frac{4-3x^3}{\sqrt{x^6+9}} &= \frac{4\left(\lim_{x\to-\infty} 1/x\right)^3 - \lim_{x\to-\infty} 3}{-\sqrt{\lim_{x\to-\infty}(1) + 9\left(\lim_{x\to-\infty} 1/x\right)^6}} \\ &= \frac{4(0)^3-(3)}{-\sqrt{(1)+9(0)^6}} \text{ by Example 2.6.1} \\ &= \frac{-3}{-1} = \boxed{3}. \qquad \square \end{aligned}$$

Exercise 2.6.68

Exercise 2.6.68. Find the horizontal asymptote(s) of the graph of $y = \dfrac{2x}{x+1}$. Justify your computations with Theorem 2.12.

Solution. By definition of horizontal asymptote, we are led to consider consider $\displaystyle\lim_{x\to\pm\infty} \frac{2x}{x+1}$. We have

$$\lim_{x\to\pm\infty} \frac{2x}{x+1} = \lim_{x\to\pm\infty} \frac{2x}{x+1}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to\pm\infty} \frac{(2x)(1/x)}{(x+1)(1/x)} = \lim_{x\to\pm\infty} \frac{(2x/x)}{(x/x+1/x)}$$

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$$= \lim_{x\to\pm\infty} \frac{2}{1+1/x}$$ since $x \to \pm\infty$ then we can assume that $x \neq 0$

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$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x}{x+1} &= \lim_{x\to\pm\infty} \frac{2x}{x+1}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator} \\
&= \lim_{x\to\pm\infty} \frac{(2x)(1/x)}{(x+1)(1/x)} = \lim_{x\to\pm\infty} \frac{(2x/x)}{(x/x+1/x)} \\
&= \lim_{x\to\pm\infty} \frac{2}{1+1/x} \text{ since } x \to \pm\infty \text{ then we can assume that } x \neq 0
\end{aligned}$$

Exercise 2.6.68 (continued)

Solution (continued).

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x}{x+1} &= \lim_{x\to\pm\infty} \frac{2}{1+1/x} \\
&= \frac{\lim_{x\to\pm\infty} 2}{\lim_{x\to\pm\infty}(1+1/x)} \text{ by the Quotient Rule} \\
&\quad \text{(Theorem 2.12(5)), assuming the denominator is not 0} \\
&= \frac{\lim_{x\to\pm\infty} 2}{\lim_{x\to\pm\infty}(1) + \lim_{x\to\pm\infty}(1/x)} \text{ by the Sum Rule,} \\
&\quad \text{Theorem 2.12(1)} \\
&= \frac{(2)}{(1)+(0)} = 2 \text{ by Example 2.6.1.}
\end{aligned}$$

Since $\lim_{x\to\pm\infty} \frac{2x}{x+1} = 2$, then $\boxed{y = 2 \text{ is a horizontal asymptote}}$ of the graph of $y = \frac{2x}{x+1}$. □

Exercise 2.6.68 (continued)

Solution (continued).

$$\lim_{x\to\pm\infty}\frac{2x}{x+1} = \lim_{x\to\pm\infty}\frac{2}{1+1/x}$$

$$= \frac{\lim_{x\to\pm\infty}2}{\lim_{x\to\pm\infty}(1+1/x)}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

$$= \frac{\lim_{x\to\pm\infty}2}{\lim_{x\to\pm\infty}(1)+\lim_{x\to\pm\infty}(1/x)}$$ by the Sum Rule, Theorem 2.12(1)

$$= \frac{(2)}{(1)+(0)} = 2$$ by Example 2.6.1.

Since $\lim_{x\to\pm\infty}\frac{2x}{x+1} = 2$, then $\boxed{y = 2 \text{ is a horizontal asymptote}}$ of the graph of $y = \frac{2x}{x+1}$. $\square$

Example 2.6.4

Example 2.6.4. Find the horizontal asymptote(s) of the graph of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$. Justify your computations with Theorem 2.12.

Solution. A rational function can have only one horizontal asymptote. Since we are not given a rational function (because of the presence of the absolute value), then we consider $x \to \infty$ and $x \to -\infty$ separately. We divide the numerator and denominator by the highest (effective) power of x in the denominator.

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$$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1}\left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to\infty} \frac{(x^3 - 2)(1/x^3)}{(|x|^3 + 1)(1/x^3)} = \lim_{x\to\infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3} \text{ since}$$

$x \to \infty$ then we can assume that x is positive so that $|x|^3 = x^3$

Example 2.6.4

Example 2.6.4. Find the horizontal asymptote(s) of the graph of $y = \dfrac{x^3 - 2}{|x|^3 + 1}$. Justify your computations with Theorem 2.12.

Solution. A rational function can have only one horizontal asymptote. Since we are not given a rational function (because of the presence of the absolute value), then we consider $x \to \infty$ and $x \to -\infty$ separately. We divide the numerator and denominator by the highest (effective) power of x in the denominator. We have

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} &= \lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1}\left(\frac{1/x^3}{1/x^3}\right) \\
&= \lim_{x\to\infty} \frac{(x^3 - 2)(1/x^3)}{(|x|^3 + 1)(1/x^3)} = \lim_{x\to\infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3} \text{ since} \\
&\quad x \to \infty \text{ then we can assume that } x \text{ is positive} \\
&\quad \text{so that } |x|^3 = x^3
\end{aligned}$$

Example 2.6.4 (continued 1)

Solution (continued).

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to\infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3}$$

$$= \lim_{x\to\infty} \frac{1-2/x^3}{1+1/x^3}$$ since $x \to \infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to\infty}(1-2/x^3)}{\lim_{x\to\infty}(1+1/x^3)}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

$$= \frac{\lim_{x\to\infty}(1) - \lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1) + \lim_{x\to\infty}(1/x^3)}$$ by the Sum Rule and the Difference Rule, Theorem 2.12(1 and 2)

Example 2.6.4 (continued 1)

Solution (continued).

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} &= \lim_{x\to\infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3} \\
&= \lim_{x\to\infty} \frac{1-2/x^3}{1+1/x^3} \text{ since } x \to \infty \text{ then we} \\
&\quad \text{can assume that } x \neq 0 \\
&= \frac{\lim_{x\to\infty}(1-2/x^3)}{\lim_{x\to\infty}(1+1/x^3)} \text{ by the Quotient Rule} \\
&\quad \text{(Theorem 2.12(5)), assuming the denominator is not 0} \\
&= \frac{\lim_{x\to\infty}(1) - \lim_{x\to\infty}(2/x^3)}{\lim_{x\to\infty}(1) + \lim_{x\to\infty}(1/x^3)} \text{ by the Sum Rule} \\
&\quad \text{and the Difference Rule, Theorem 2.12(1 and 2)}
\end{aligned}$$

Example 2.6.4 (continued 2)

Solution (continued).

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \frac{\lim_{x\to\infty}(1) - 2\left(\lim_{x\to\infty}(1/x)\right)^3}{\lim_{x\to\infty}(1) + \left(\lim_{x\to\infty}(1/x)\right)^3}$$ by the Constant Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)

$$= \frac{(1) - 2(0)^3}{(1) + (0)^3} = 1$$ by Example 2.6.1(a).

So the graph of $y = \dfrac{x^3-2}{|x|^3+1}$ has a horizontal asymptote of $y = 1$ as $x \to \infty$.

Example 2.6.4 (continued 2)

Solution (continued).

$$\lim_{x\to\infty} \frac{x^3-2}{|x|^3+1} = \frac{\lim_{x\to\infty}(1) - 2\left(\lim_{x\to\infty}(1/x)\right)^3}{\lim_{x\to\infty}(1) + \left(\lim_{x\to\infty}(1/x)\right)^3}$$ by the Constant Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)

$$= \frac{(1)-2(0)^3}{(1)+(0)^3} = 1$$ by Example 2.6.1(a).

So the graph of $y = \dfrac{x^3-2}{|x|^3+1}$ has a

horizontal asymptote of $y = 1$ as $x \to \infty$.

Example 2.6.4 (continued 3)

Solution (continued). The computation is similar for $x \to -\infty$, except that for x negative we have $|x|^3 = -x^3$. We have

$$\lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1} = \lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1}\left(\frac{1/x^3}{1/x^3}\right)$$

$$= \lim_{x\to-\infty} \frac{(x^3-2)(1/x^3)}{(|x|^3+1)(1/x^3)} = \lim_{x\to-\infty} \frac{x^3/x^3-2/x^3}{-x^3/x^3+1/x^3}$$ since $x \to -\infty$ then we can assume that x is negative so that $|x|^3 = -x^3$

$$= \lim_{x\to-\infty} \frac{1-2/x^3}{-1+1/x^3}$$ since $x \to -\infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to-\infty}(1-2/x^3)}{\lim_{x\to-\infty}(-1+1/x^3)}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

Example 2.6.4 (continued 3)

Solution (continued). The computation is similar for $x \to -\infty$, except that for x negative we have $|x|^3 = -x^3$. We have

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1} &= \lim_{x\to-\infty} \frac{x^3-2}{|x|^3+1}\left(\frac{1/x^3}{1/x^3}\right) \\
&= \lim_{x\to-\infty} \frac{(x^3-2)(1/x^3)}{(|x|^3+1)(1/x^3)} = \lim_{x\to-\infty} \frac{x^3/x^3-2/x^3}{-x^3/x^3+1/x^3} \text{ since } \\
&\quad x \to -\infty \text{ then we can assume that } x \text{ is negative} \\
&\quad \text{so that } |x|^3 = -x^3 \\
&= \lim_{x\to-\infty} \frac{1-2/x^3}{-1+1/x^3} \text{ since } x \to -\infty \text{ then we} \\
&\quad \text{can assume that } x \neq 0 \\
&= \frac{\lim_{x\to-\infty}(1-2/x^3)}{\lim_{x\to-\infty}(-1+1/x^3)} \text{ by the Quotient Rule} \\
&\quad \text{(Theorem 2.12(5)), assuming the denominator is not 0}
\end{aligned}$$

Example 2.6.4 (continued 4)

Solution (continued).

$$\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} = \frac{\lim_{x\to-\infty}(1)-\lim_{x\to-\infty}(2/x^3)}{\lim_{x\to-\infty}(-1)+\lim_{x\to-\infty}(1/x^3)}$$ by the Sum Rule and the Difference Rule, Theorem 2.12(1 and 2)

$$= \frac{\lim_{x\to-\infty}(1)-2\left(\lim_{x\to-\infty}(1/x)\right)^3}{\lim_{x\to-\infty}(-1)+\left(\lim_{x\to-\infty}(1/x)\right)^3}$$ by Const. Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)

$$= \frac{(1)-2(0)^3}{(-1)+(0)^3} = -1$$ by Example 2.6.1(b).

So the graph of $y=\dfrac{x^3-2}{|x|^3+1}$ has a horizontal asymptote of $y=-1$ as $x\to-\infty$. □

Example 2.6.4 (continued 4)

Solution (continued).

$$\lim_{x\to-\infty}\frac{x^3-2}{|x|^3+1} = \frac{\lim_{x\to-\infty}(1)-\lim_{x\to-\infty}(2/x^3)}{\lim_{x\to-\infty}(-1)+\lim_{x\to-\infty}(1/x^3)} \text{ by the Sum Rule}$$

and the Difference Rule, Theorem 2.12(1 and 2)

$$= \frac{\lim_{x\to-\infty}(1)-2\left(\lim_{x\to-\infty}(1/x)\right)^3}{\lim_{x\to-\infty}(-1)+\left(\lim_{x\to-\infty}(1/x)\right)^3} \text{ by Const. Mult.}$$

Rule and the Power Rule, Theorem 2.12(4 and 6)

$$= \frac{(1)-2(0)^3}{(-1)+(0)^3} = -1 \text{ by Example 2.6.1(b).}$$

So the graph of $y = \dfrac{x^3-2}{|x|^3+1}$ has a

$\boxed{\text{horizontal asymptote of } y=-1 \text{ as } x\to-\infty}$. □

Example 2.6.5

Example 2.6.5. Use the formal definition to prove $\lim_{x\to-\infty} e^x = 0$. Notice that this implies that $y = 0$ is a horizontal asymptote of $y = e^x$.

Proof. First, the domain of $f(x) = e^x$ is all of the real numbers $\mathbb{R}$, so it is defined on an interval of the form $(-\infty, P)$ (for any P). Next, let $\varepsilon > 0$. Choose $N = \ln \varepsilon$.

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That is, if $x < N$ then

$|f(x) - 0| = |e^x - 0| = e^x < \varepsilon$.

Therefore, by definition,

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^x = 0$,

as claimed. $\square$

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Therefore, by definition,
$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^x = 0$,
as claimed. □

Note. The choice of $N = \ln \varepsilon$ makes sense if we consider the graph of $y = e^x$:

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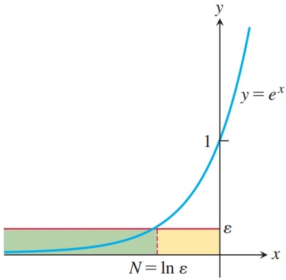

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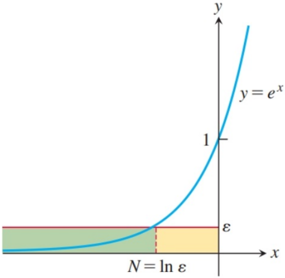

Example 2.6.A

Example 2.6.A. Evaluate $\lim_{x\to\infty}\cos(1/x)$.

Solution. By Example 2.6.1(a) we have $\lim_{x\to\infty} 1/x = 0$, and by Exercise 2.5.72 we have that $\cos x$ is continuous at all points (in particular, it is continuous at 0). So by Theorem 2.6.A,

$$\lim_{x\to\infty}\cos(1/x) = \cos\left(\lim_{x\to\infty} 1/x\right) = \cos 0 = \boxed{1}.$$

□

Example 2.6.A

Example 2.6.A. Evaluate $\lim_{x\to\infty} \cos(1/x)$.

Solution. By Example 2.6.1(a) we have $\lim_{x\to\infty} 1/x = 0$, and by Exercise 2.5.72 we have that $\cos x$ is continuous at all points (in particular, it is continuous at 0). So by Theorem 2.6.A,

$$\lim_{x\to\infty} \cos(1/x) = \cos\left(\lim_{x\to\infty} 1/x\right) = \cos 0 = \boxed{1}.$$

□

Example 2.6.8

Example 2.6.8. Use the Sandwich Theorem to find the horizontal asymptote of the curve $y = 2 + \dfrac{\sin x}{x}$.

Solution. First, $-1 \le \sin x \le 1$ for all real numbers. Let $g(x) = 2 - 1/x$, $f(x) = 2 + \dfrac{\sin x}{x}$, and $h(x) = 2 + 1/x$. Then $g(x) \le f(x) \le h(x)$ for all real numbers, except 0, and so these inequalities hold on $(-\infty, P) = (-\infty, 0)$ and $(P, \infty) = (0, \infty)$.

Example 2.6.8

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Example 2.6.8

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Solution. First, $-1 \le \sin x \le 1$ for all real numbers. Let $g(x) = 2 - 1/x$, $f(x) = 2 + \dfrac{\sin x}{x}$, and $h(x) = 2 + 1/x$. Then $g(x) \le f(x) \le h(x)$ for all real numbers, except 0, and so these inequalities hold on $(-\infty, P) = (-\infty, 0)$ and $(P, \infty) = (0, \infty)$. Now $\lim_{x \to \pm\infty} g(x) = \lim_{x \to \pm\infty} (2 - 1/x) = 2 - (0) = 2 = L$ and $\lim_{x \to \pm\infty} h(x) = \lim_{x \to \pm\infty} (2 + 1/x) = 2 + (0) = 2 = L$, by Example 2.6.1. So by Theorem 2.6.B, $\displaystyle\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} 2 + \frac{\sin x}{x} = 2$. Therefore, $\boxed{y = 2 \text{ is a horizontal asymptote}}$ of the graph of $y = 2 + \dfrac{\sin x}{x}$. $\square$

Exercise 2.6.92

Exercise 2.6.92. Evaluate (carefully!) $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$. Justify your computations.

Solution. We multiply by the conjugate of the given expression divided by itself (which is defined for x "sufficiently large," namely $x \geq 1$) in order to produce a quotient and try to use some of the techniques already introduced.

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Solution. We multiply by the conjugate of the given expression divided by itself (which is defined for x "sufficiently large," namely $x \geq 1$) in order to produce a quotient and try to use some of the techniques already introduced. We have

$$\begin{aligned}
&\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\\
=&\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\left(\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)\\
=&\lim_{x\to\infty}\frac{(\sqrt{x^2+x})^2-(\sqrt{x^2-x})^2}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
=&\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}
\end{aligned}$$

Exercise 2.6.92

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Solution. We multiply by the conjugate of the given expression divided by itself (which is defined for x "sufficiently large," namely $x \geq 1$) in order to produce a quotient and try to use some of the techniques already introduced. We have

$$\begin{aligned}
&\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\\
=\;&\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\left(\frac{\sqrt{x^2+x}+\sqrt{x^2-x}}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)\\
=\;&\lim_{x\to\infty}\frac{(\sqrt{x^2+x})^2-(\sqrt{x^2-x})^2}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}\\
=\;&\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}
\end{aligned}$$

Exercise 2.6.92 (continued 1)

Solution (continued).

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}} \left(\frac{1/x}{1/x}\right)$$ dividing the numerator and denominator by the effective highest power of x in the denominator

$$= \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/x} = \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/\sqrt{x^2}}$$

since $x \to \infty$ then we can assume that x is positive so that $\sqrt{x^2} = |x| = x$

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2+x}/\sqrt{x^2}+\sqrt{x^2-x}/\sqrt{x^2}}$$

Exercise 2.6.92 (continued 1)

Solution (continued).

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}$$

$$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}\left(\frac{1/x}{1/x}\right)$$ dividing the numerator and denominator by the effective highest power

of x in the denominator

$$= \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/x} = \lim_{x\to\infty} \frac{(2x)/x}{(\sqrt{x^2+x}+\sqrt{x^2-x})/\sqrt{x^2}}$$

since $x \to \infty$ then we can assume that x is positive

so that $\sqrt{x^2} = |x| = x$

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2+x}/\sqrt{x^2}+\sqrt{x^2-x}/\sqrt{x^2}}$$

Exercise 2.6.92 (continued 2)

Solution (continued).

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{(x^2+x)/x^2}+\sqrt{(x^2-x)/x^2}}$$

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2/x^2+x/x^2}+\sqrt{x^2/x^2-x/x^2}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}$$ since $x\to\infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}(\sqrt{1+1/x}+\sqrt{1-1/x})}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}\sqrt{1+1/x}+\lim_{x\to\infty}\sqrt{1-1/x})}$$ by the Sum Rule, Theorem 2.12(1)

Exercise 2.6.92 (continued 2)

Solution (continued).

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{(x^2+x)/x^2}+\sqrt{(x^2-x)/x^2}}$$

$$= \lim_{x\to\infty} \frac{(2x)/x}{\sqrt{x^2/x^2+x/x^2}+\sqrt{x^2/x^2-x/x^2}}$$

$$= \lim_{x\to\infty} \frac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}$$ since $x \to \infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}(\sqrt{1+1/x}+\sqrt{1-1/x})}$$ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}\sqrt{1+1/x}+\lim_{x\to\infty}\sqrt{1-1/x})}$$ by the Sum Rule, Theorem 2.12(1)

Exercise 2.6.92 (continued 3)

Solution (continued).

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty} \sqrt{1+1/x} + \lim_{x\to\infty} \sqrt{1-1/x})}$$

$$= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty}(1+1/x)} + \sqrt{\lim_{x\to\infty}(1-1/x)})}$$ by the Root Rule, Theorem 2.12(7) (notice that both $1+1/x$ and $1-1/x$ are nonnegative for $x \geq 1$)

$$= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(1/x)} + \sqrt{\lim_{x\to\infty} 1 - \lim_{x\to\infty}(1/x)})}$$

by the Sum and Difference Rules, Theorem 2.12(1 and 2)

$$= \frac{(2)}{\sqrt{(1)+(0)} + \sqrt{(1)-(0)})} = \frac{2}{1+1} = \boxed{1}. \qquad \square$$

Exercise 2.6.92 (continued 3)

Solution (continued).

$$= \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty}\sqrt{1+1/x} + \lim_{x\to\infty}\sqrt{1-1/x})}$$

$$= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty}(1+1/x)} + \sqrt{\lim_{x\to\infty}(1-1/x)})}$$ by the Root Rule, Theorem 2.12(7) (notice that both $1+1/x$ and $1-1/x$ are nonnegative for $x \geq 1$)

$$= \frac{\lim_{x\to\infty} 2}{\sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty}(1/x)} + \sqrt{\lim_{x\to\infty} 1 - \lim_{x\to\infty}(1/x)})}$$ by the Sum and Difference Rules, Theorem 2.12(1 and 2)

$$= \frac{(2)}{\sqrt{(1)+(0)} + \sqrt{(1)-(0)})} = \frac{2}{1+1} = \boxed{1}. \qquad \square$$

Exercise 2.6.108

Exercise 2.6.108. Consider the rational function $y = \dfrac{x^2 - 1}{2x + 4}$. Find the oblique asymptote.

Solution. First, we perform long division to get:

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$$
\begin{array}{r|rrrrr}
 & & & x/2 & - & 1 \\
\hline
2x+4 \,) & x^2 & & & - & 1 \\
 & x^2 & + & 2x & & \\
\hline
 & & - & 2x & - & 1 \\
 & & - & 2x & - & 4 \\
\hline
 & & & & & 3
\end{array}
$$

So $y = \dfrac{x^2 - 1}{2x + 4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x + 4}$ where $x/2 - 1$ is a linear term. If we show that $\lim_{x \to \pm\infty} 3/(2x + 4) = 0$ then we can conclude that $y = x/2 - 1$ is the oblique asymptote for the graph of $y = (x^2 - 1)/(2x + 4)$.

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Exercise 2.6.108 (continued 1)

Solution (continued). Next,

$$\lim_{x\to\pm\infty} \frac{3}{2x+4} = \lim_{x\to\pm\infty} \frac{3}{2x+4}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to\pm\infty} \frac{(3)(1/x)}{(2x+4)(1/x)} = \lim_{x\to\pm\infty} \frac{3/x}{2x/x+4/x}$$

$$= \lim_{x\to\pm\infty} \frac{3/x}{2+4/x} \text{ since } x\to\pm\infty \text{ then we can assume that } x\neq 0$$

$$= \frac{3\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 2 + 4\lim_{x\to\pm\infty} 1/x} \text{ by the Sum, Constant, Multiple and Quotient Rules, Theorem 2.12(1, 4, \& 5)}$$

$$= \frac{3(0)}{(2)+4(0)} = 0 \text{ by Example 2.6.1.}$$

Exercise 2.6.108 (continued 2)

Exercise 2.6.108. Consider the rational function $y = \dfrac{x^2-1}{2x+4}$. Find the oblique asymptote.

Solution (continued). Since $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$ and $\lim_{x\to\pm\infty} \dfrac{3}{2x+4} = 0$, then $\boxed{y = \dfrac{x}{2} - 1 \text{ is an oblique asymptote}}$ of the graph of $y = \dfrac{x^2-1}{2x+4}$. Notice that the function $f(x) = \dfrac{x^2-1}{2x+4}$ is not defined at $x = -2$. With $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$, the term $\dfrac{3}{2x+4}$ is positive for x large and positive, and is negative for x large and negative. So the graph of $y = \dfrac{x^2-1}{2x+4}$ lies above the oblique asymptote $y = \dfrac{x}{2} - 1$ for x large and positive, and lies below the oblique asymptote for x large and negative.

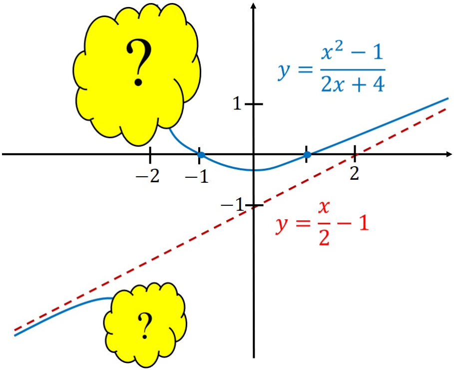
Exercise 2.6.108 (continued 2)

Exercise 2.6.108. Consider the rational function $y = \dfrac{x^2-1}{2x+4}$. Find the oblique asymptote.

Solution (continued). Since $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$ and $\lim_{x \to \pm\infty} \dfrac{3}{2x+4} = 0$, then $\boxed{y = \dfrac{x}{2} - 1 \text{ is an oblique asymptote}}$ of the graph of $y = \dfrac{x^2-1}{2x+4}$. Notice that the function $f(x) = \dfrac{x^2-1}{2x+4}$ is not defined at $x = -2$. With $y = \dfrac{x^2-1}{2x+4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x+4}$, the term $\dfrac{3}{2x+4}$ is positive for x large and positive, and is negative for x large and negative. So the graph of $y = \dfrac{x^2-1}{2x+4}$ lies above the oblique asymptote $y = \dfrac{x}{2} - 1$ for x large and positive, and lies below the oblique asymptote for x large and negative.

Exercise 2.6.108 (continued 3)

Solution (continued). A crude graph of $y = \dfrac{x^2-1}{2x+4}$ which reflects the oblique asymptote (but does not reflect other subtle details of the graph) is as follows (we'll explore the graph in more detail later):



Exercise 2.6.108 (continued 3)

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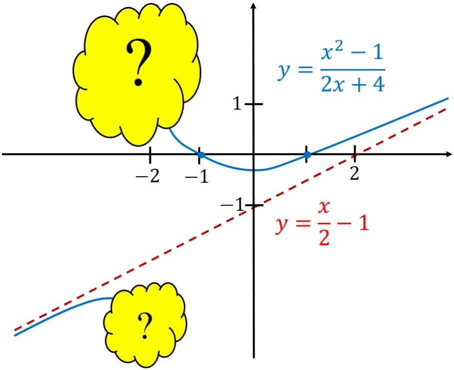

□

Example 2.6.B

Example 2.6.B. For n a positive even integer, prove that $\lim_{x \to 0} \frac{1}{x^n} = \infty$.

Solution. First, $f(x) = 1/x^n$ is defined for all x except 0, so there is an open interval containing $c = 0$ on which f is defined, except at $c = 0$ itself (say the interval $(-1, 1)$). Let B be a positive real number. Choose $\delta = 1/B^{1/n}$.

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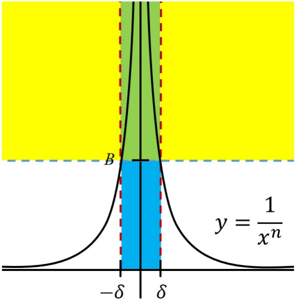

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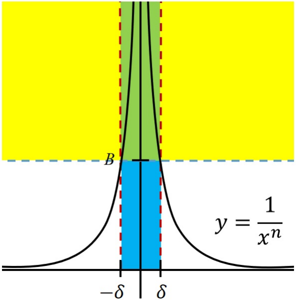

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Exercise 2.6.54

Exercise 2.6.54. Consider $f(x) = \dfrac{x}{x^2 - 1}$. Find (a) $\lim_{x \to 1^+} f(x)$, (b) $\lim_{x \to 1^-} f(x)$, (c) $\lim_{x \to -1^+} f(x)$, and (d) $\lim_{x \to -1^-} f(x)$.

Solution. First, $f(x) = \dfrac{x}{x^2 - 1}$ is a rational function of the form $f(x) = p(x)/q(x)$ where $p(x) = x$ and $q(x) = x^2 - 1$.

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(a) We have $\lim_{x \to 1^+} p(x) = \lim_{x \to 1^+} x = 1 \neq 0$ and $\lim_{x \to 1^+} q(x) = \lim_{x \to 1^+}(x^2 - 1) = (1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x \to 1^+} \frac{p(x)}{q(x)} = \lim_{x \to 1^+} \frac{x}{x^2 - 1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$.

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Exercise 2.6.54 (continued 1)

Solution (continued). Combining the factors we can conclude the following little sign diagram (not an actual equation):
$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. Since we know $\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly greater than 1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \infty}$. □

(b) We have $\lim_{x\to 1^-} p(x) = \lim_{x\to 1^-} x = 1 \neq 0$ and $\lim_{x\to 1^-} q(x) = \lim_{x\to 1^-}(x^2-1) = (1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to 1^-} \frac{p(x)}{q(x)} = \lim_{x\to 1^-} \frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$.

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Solution (continued). Combining the factors we can conclude the following little sign diagram (not an actual equation):
$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. Since we know $\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly greater than 1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \infty}$. □

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Exercise 2.6.54 (continued 1)

Solution (continued). Combining the factors we can conclude the following little sign diagram (not an actual equation):
$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. Since we know
$\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly greater than 1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^+} \frac{x}{x^2-1} = \infty}$. $\square$

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$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to 1^-$, then appropriate x are close to 1 and slightly less than 1).

Exercise 2.6.54 (continued 2)

Solution (continued). For such x, we have x is positive (in fact, x is "close to" 1), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive (in fact, x is "close to" 2). Combining the factors we can again conclude the following little sign diagram: $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. Since we know $\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly less than 1 that $\dfrac{x}{x^2-1}$ is negative, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty}$. □

(c) We have $\lim_{x\to -1^+} p(x) = \lim_{x\to -1^+} x = -1 \neq 0$ and $\lim_{x\to -1^+} q(x) = \lim_{x\to -1^+}(x^2-1) = (-1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem, $\displaystyle\lim_{x\to -1^+} \frac{p(x)}{q(x)} = \lim_{x\to -1^+} \frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$.

Exercise 2.6.54 (continued 2)

Solution (continued). For such x, we have x is positive (in fact, x is "close to" 1), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive (in fact, x is "close to" 2). Combining the factors we can again conclude the following little sign diagram: $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. Since we know $\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to 1 and slightly less than 1 that $\dfrac{x}{x^2-1}$ is negative, then we conclude that $\boxed{\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = -\infty}$. □

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Exercise 2.6.54 (continued 3)

Solution (continued). We analyze the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to -1^+$, then appropriate x are close to -1 and slightly greater than -1). For such x, we have x is negative (in fact, x is "close to" -1), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:

$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. Since we know $\displaystyle\lim_{x\to 1^-} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to -1 and slightly greater than -1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that

$$\boxed{\lim_{x\to -1^+} \frac{x}{x^2-1} = \infty}. \quad \square$$

Exercise 2.6.54 (continued 3)

Solution (continued). We analyze the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to -1^+$, then appropriate x are close to -1 and slightly greater than -1). For such x, we have x is negative (in fact, x is "close to" -1), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:

$\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. Since we know

$\lim\limits_{x\to 1^-} \dfrac{x}{x^2-1} = \pm\infty$ and we know for x close to -1 and slightly greater than -1 that $\dfrac{x}{x^2-1}$ is positive, then we conclude that

$\boxed{\lim\limits_{x\to -1^+} \dfrac{x}{x^2-1} = \infty}$. □

Exercise 2.6.54 (continued 4)

Solution (continued). **(d)** We have

$\lim_{x\to -1^-} p(x) = \lim_{x\to -1^-} x = -1 \neq 0$ and
$\lim_{x\to -1^-} q(x) = \lim_{x\to -1^-}(x^2 - 1) = (-1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem,
$\displaystyle\lim_{x\to -1^-} \frac{p(x)}{q(x)} = \lim_{x\to -1^-} \frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$.

Exercise 2.6.54 (continued 4)

Solution (continued). (d) We have

$\lim_{x\to -1^-} p(x) = \lim_{x\to -1^-} x = -1 \neq 0$ and
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$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} \Rightarrow \frac{(-)}{(-)(-)} = -.$$

Exercise 2.6.54 (continued 4)

Solution (continued). (d) We have

$\lim_{x\to -1^-} p(x) = \lim_{x\to -1^-} x = -1 \neq 0$ and
$\lim_{x\to -1^-} q(x) = \lim_{x\to -1^-}(x^2-1) = (-1)^2 - 1 = 0$, by Theorem 2.2 for one-sided limits. So by Dr. Bob's Infinite Limits Theorem,
$\displaystyle\lim_{x\to -1^-} \frac{p(x)}{q(x)} = \lim_{x\to -1^-} \frac{x}{x^2-1} = \pm\infty$; we just need to determine if the limit is $+\infty$ or $-\infty$. We analyze the sign of $\dfrac{x}{x^2-1} = \dfrac{x}{(x-1)(x+1)}$ for "appropriate" x (since $x \to -1^-$, then appropriate x are close to -1 and slightly less than -1). For such x, we have x is negative (in fact, x is "close to" -1), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is negative (since x is less than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:

$$\frac{x}{x^2-1} = \frac{x}{(x-1)(x+1)} \Rightarrow \frac{(-)}{(-)(-)} = -.$$

Exercise 2.6.54 (continued 5)

Solution (continued). Since we know $\lim_{x \to 1^-} \dfrac{x}{x^2 - 1} = \pm\infty$ and we know for x close to -1 and slightly less than -1 that $\dfrac{x}{x^2 - 1}$ is negative, then we conclude that $\boxed{\lim_{x \to -1^-} \dfrac{x}{x^2 - 1} = -\infty}$. □

Note. We know a lot about $f(x) = \dfrac{x}{x^2 - 1}$ and can get a reasonable graph of $y = f(x)$ by graphing its vertical asymptotes (notice also that $f(0) = 0$; we did not explore what happens for $|x|$ "large"):

Exercise 2.6.54 (continued 5)

Solution (continued). Since we know $\lim_{x \to 1^-} \frac{x}{x^2-1} = \pm\infty$ and we know for x close to -1 and slightly less than -1 that $\frac{x}{x^2-1}$ is negative, then we conclude that $\boxed{\lim_{x \to -1^-} \frac{x}{x^2-1} = -\infty}$. $\square$

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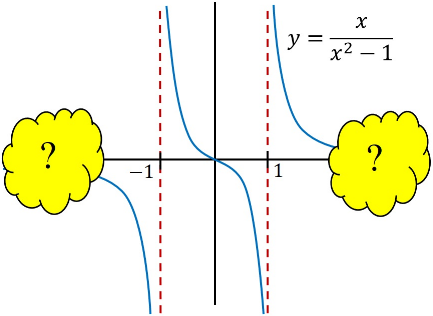

Exercise 2.6.54 (continued 5)

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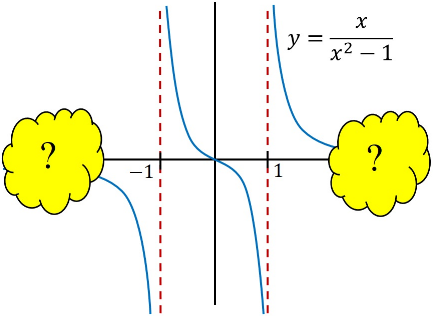
Note. We know a lot about $f(x) = \frac{x}{x^2-1}$ and can get a reasonable graph of $y = f(x)$ by graphing its vertical asymptotes (notice also that $f(0) = 0$; we did not explore what happens for $|x|$ "large"):



Exercise 2.6.70

Exercise 2.6.70. Consider $y = f(x) = \dfrac{2x}{x^2 - 1}$. Find the domain, horizontal asymptote(s), vertical asymptotes, graph $y = f(x)$ in such a way as to reflect the asymptotic behavior, and find the range of f.

Solution. First, the domain of $y = f(x) = \dfrac{2x}{x^2 - 1} = \dfrac{2x}{(x+1)(x-1)}$ is all real x except for -1 and 1; the $\boxed{\text{domain is } (-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$.

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For the horizontal asymptote(s), we consider $\lim_{x\to\pm\infty} f(x)$. We have

$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty}\frac{2x}{x^2-1}\\ &= \lim_{x\to\pm\infty}\frac{2x}{x^2-1}\left(\frac{1/x^2}{1/x^2}\right)\end{aligned}$$

dividing the numerator and denominator by the highest power of x in the denominator

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$$\begin{aligned}\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{2x}{x^2-1} \\ &= \lim_{x\to\pm\infty} \frac{2x}{x^2-1}\left(\frac{1/x^2}{1/x^2}\right) \text{ dividing the numerator and} \\ &\quad \text{denominator by the highest power} \\ &\quad \text{of } x \text{ in the denominator}\end{aligned}$$

Exercise 2.6.70 (continued 1)

Solution (continued).

$$\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty} \frac{(2x)/x^2}{(x^2-1)/x^2} = \lim_{x\to\pm\infty} \frac{2/x}{1-1/x^2} \text{ since } x \to \infty \\
&\quad \text{then we can assume that } x \neq 0 \\
&= \frac{\lim_{x\to\pm\infty} 2/x}{\lim_{x\to\pm\infty}(1-1/x^2)} \text{ by the Quotient Rule} \\
&\quad \text{(Theorem 2.12(5)), assuming the denominator is not 0} \\
&= \frac{2\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 1 - (\lim_{x\to\pm\infty} 1/x)^2} \text{ by the Difference,} \\
&\quad \text{Constant Mult., and Power Rules, Theorem 2.12(2, 4, 6)} \\
&= \frac{2(0)}{1-(0)^2} = \frac{0}{1} = 0 \text{ by Example 2.6.1.}
\end{aligned}$$

So $\boxed{y = 0 \text{ is a horizontal asymptote}}$ of $y = \dfrac{2x}{x^2-1}$.

Exercise 2.6.70 (continued 2)

Solution (continued). Now $f(x) = \dfrac{2x}{x^2-1} = \dfrac{2x}{(x+1)(x-1)}$ is a rational function with $\lim_{x\to-1} 2x = -2 \neq 0$, $\lim_{x\to-1} x^2-1 = 0$, and $\lim_{x\to 1} x^2-1 = 0$ (each by Theorem 2.2), so by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\boxed{f \text{ has vertical asymptotes at } x=-1 \text{ and } x=1}$. We explore the vertical asymptotes by taking one-sided limits to determine if the limit is $+\infty$ or $-\infty$. We analyze the sign of $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)}$ for "appropriate" x in each case. For $x \to -1^+$, the appropriate x are close to -1 and slightly greater than -1. For such x, we have $2x$ is negative (in fact, $2x$ is "close to" -2), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:
$\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. So $\boxed{\lim_{x\to-1^+} f(x) = \infty}$.

Exercise 2.6.70 (continued 2)

Solution (continued). Now $f(x) = \dfrac{2x}{x^2-1} = \dfrac{2x}{(x+1)(x-1)}$ is a rational function with $\lim_{x\to -1} 2x = -2 \neq 0$, $\lim_{x\to -1} x^2 - 1 = 0$, and $\lim_{x\to 1} x^2 - 1 = 0$ (each by Theorem 2.2), so by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\boxed{f \text{ has vertical asymptotes at } x = -1 \text{ and } x = 1}$. We explore the vertical asymptotes by taking one-sided limits to determine if the limit is $+\infty$ or $-\infty$. We analyze the sign of $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)}$ for "appropriate" x in each case. For $x \to -1^+$, the appropriate x are close to -1 and slightly greater than -1. For such x, we have $2x$ is negative (in fact, $2x$ is "close to" -2), $x-1$ is negative (in fact, $x-1$ is "close to" -2), and $x+1$ is positive (since x is greater than -1; so $x+1$ is positive and "close to" 0). Combining the factors we get the sign diagram:

$\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(+)} = +$. So $\boxed{\lim_{x\to -1^+} f(x) = \infty}$.

Exercise 2.6.70 (continued 3)

Solution (continued). For $x \to -1^-$, the appropriate x are close to -1 and slightly less than -1. For such x, we have $2x$ is negative ($2x$ is "close to" -2), $x-1$ is negative ($x-1$ is "close to" -2), and $x+1$ is negative (since x is less than -1; so $x+1$ is negative and "close to" 0). Combining the factors we get the sign diagram:

$$\frac{2x}{x^2-1} = \frac{2x}{(x-1)(x+1)} \Rightarrow \frac{(-)}{(-)(-)} = -.$$ So $\boxed{\lim_{x\to -1^-} f(x) = -\infty}$.

For $x \to 1^+$, the appropriate x are close to 1 and slightly greater than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is positive (since x is greater than 1; so $x-1$ is positive and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. So $\boxed{\lim_{x\to 1^+} f(x) = \infty}$.

Exercise 2.6.70 (continued 3)

Solution (continued). For $x \to -1^-$, the appropriate x are close to -1 and slightly less than -1. For such x, we have $2x$ is negative ($2x$ is "close to" -2), $x-1$ is negative ($x-1$ is "close to" -2), and $x+1$ is negative (since x is less than -1; so $x+1$ is negative and "close to" 0). Combining the factors we get the sign diagram:
$\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(-)}{(-)(-)} = -$. So $\boxed{\lim_{x\to -1^-} f(x) = -\infty}$.
For $x \to 1^+$, the appropriate x are close to 1 and slightly greater than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is positive (since x is greater than 1; so $x-1$ is positive and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(+)(+)} = +$. So
$\boxed{\lim_{x\to 1^+} f(x) = \infty}$.

Exercise 2.6.70 (continued 4)

Solution (continued). For $x \to 1^-$, the appropriate x are close to 1 and slightly less than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. So $\boxed{\lim_{x\to 1^-} f(x) = -\infty}$.

We have the graph (notice the range is all real numbers):

Exercise 2.6.70 (continued 4)

Solution (continued). For $x \to 1^-$, the appropriate x are close to 1 and slightly less than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x - 1$ is negative (since x is less than 1; so $x - 1$ is negative and "close to" 0), and $x + 1$ is positive ($x + 1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. So $\boxed{\lim_{x\to 1^-} f(x) = -\infty}$.

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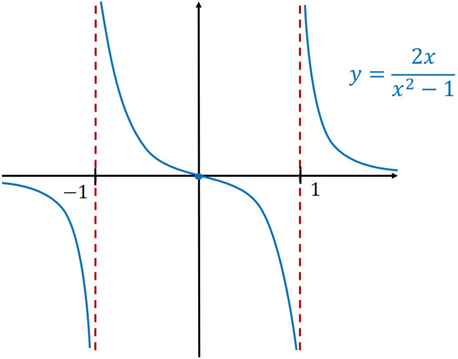

Exercise 2.6.70 (continued 4)

Solution (continued). For $x \to 1^-$, the appropriate x are close to 1 and slightly less than 1. For such x, we have $2x$ is positive ($2x$ is "close to" 2), $x-1$ is negative (since x is less than 1; so $x-1$ is negative and "close to" 0), and $x+1$ is positive ($x+1$ is "close to" 2). Combining the factors we get the sign diagram: $\dfrac{2x}{x^2-1} = \dfrac{2x}{(x-1)(x+1)} \Rightarrow \dfrac{(+)}{(-)(+)} = -$. So $\boxed{\lim_{x\to 1^-} f(x) = -\infty}$.

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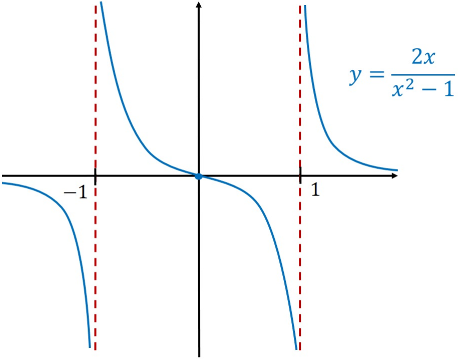

Example 2.6.20

Example 2.6.20. Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that g is a dominant term of f.

Solution. We need to show that $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 1$. We have

$$\begin{aligned}
\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \to \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\
&= \lim_{x \to \pm\infty} \left(\frac{3x^4}{3x^4} - \frac{2x^3}{3x^4} + \frac{3x^2}{3x^4} - \frac{5x}{3x^4} + \frac{6}{3x^4} \right) \\
&= \lim_{x \to \pm\infty} \left(1 - \frac{2}{3x} + \frac{3}{3x^2} - \frac{5}{3x^3} + \frac{6}{3x^4} \right) \text{ since } x \to \infty \\
&\quad \text{then we can assume that } x \neq 0
\end{aligned}$$

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&= \lim_{x\to\pm\infty} \left(\frac{3x^4}{3x^4} - \frac{2x^3}{3x^4} + \frac{3x^2}{3x^4} - \frac{5x}{3x^4} + \frac{6}{3x^4} \right) \\
&= \lim_{x\to\pm\infty} \left(1 - \frac{2}{3x} + \frac{3}{3x^2} - \frac{5}{3x^3} + \frac{6}{3x^4} \right) \text{ since } x \to \infty \\
&\quad \text{then we can assume that } x \neq 0
\end{aligned}$$

Example 2.6.20 (continued)

Solution (continued).

$$\begin{aligned}
\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\pm\infty}\left(1-\frac{2}{3x}+\frac{3}{3x^2}-\frac{5}{3x^3}+\frac{6}{3x^4}\right)\\
&= \lim_{x\to\pm\infty}1-\frac{2}{3}\lim_{x\to\pm\infty}\frac{1}{x}+\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^2-\frac{5}{3}\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^3\\
&\quad +2\left(\lim_{x\to\pm\infty}\frac{1}{x}\right)^4 \text{ by the Sum, Difference,}\\
&\quad \text{Constant Multiple, and Power Rules,}\\
&\quad \text{Theorem 2.12(1, 2, 4, 6)}\\
&= 1-\frac{2}{3}(0)+(0)^2-\frac{5}{3}(0)^3+2(0)^4=1 \text{ by Example 2.6.1.}
\end{aligned}$$

Since the limit is 1, then g is a dominant term of f, as claimed. □

Exercise 2.6.108 (again)

Exercise 2.6.108 (again). Consider the rational function $y = \dfrac{x^2 - 1}{2x + 4}$. Find all asymptotes and graph in a way that reflects the asymptotic behavior.

Solution. We saw above that the graph of $y = \dfrac{x^2 - 1}{2x + 4} = \dfrac{x}{2} - 1 + \dfrac{3}{2x + 4}$ has $y = \dfrac{x}{2} - 1$ as an oblique asymptote as $x \to \pm\infty$. We now explore vertical asymptotes.

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Exercise 2.6.108 (again, continued 1)

Solution (continued). For $\lim_{x\to -2^+} f(x)$, we analyze the sign of $\dfrac{x^2-1}{2x+4}$ for "appropriate" x (since $x \to -2^+$, then appropriate x are close to -2 and slightly greater than -2). For such x, we have x^2-1 is positive (in fact, x^2-1 is "close to" 3) and $2x+4$ is positive (since x is greater than -2; so $2x+4$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(+)} = +$. So

$$\boxed{\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2-1}{2x+4} = \infty}.$$

For $\lim_{x\to -2^-} f(x)$, we analyze the sign of $\dfrac{x^2-1}{2x+4}$ for "appropriate" x (since $x \to -2^-$, then appropriate x are close to -2 and slightly less than -2). For such x, we have x^2-1 is positive (in fact, x^2-1 is "close to" 3) and $2x+4$ is negative (since x is less than -2; so $2x+4$ is negative and "close to" 0).

Exercise 2.6.108 (again, continued 1)

Solution (continued). For $\lim_{x \to -2^+} f(x)$, we analyze the sign of $\dfrac{x^2-1}{2x+4}$ for "appropriate" x (since $x \to -2^+$, then appropriate x are close to -2 and slightly greater than -2). For such x, we have x^2-1 is positive (in fact, x^2-1 is "close to" 3) and $2x+4$ is positive (since x is greater than -2; so $2x+4$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(+)} = +$. So

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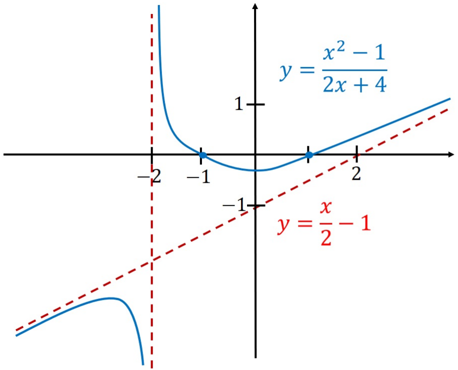
Exercise 2.6.108 (again, continued 2)

Solution (continued). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(-)} = -$. So $\boxed{\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \dfrac{x^2-1}{2x+4} = -\infty}$. So the graph is:

Exercise 2.6.108 (again, continued 2)

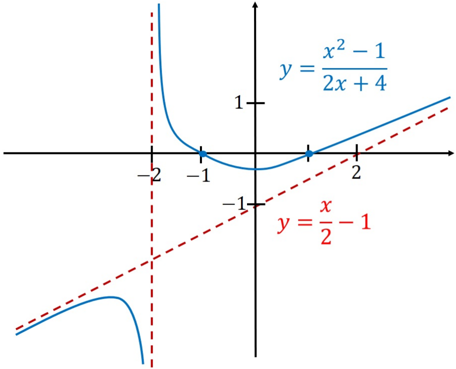
Solution (continued). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(-)} = -$. So $\boxed{\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \dfrac{x^2-1}{2x+4} = -\infty}$. So the graph is:



□

Exercise 2.6.108 (again, continued 2)

Solution (continued). Combining the factors we get the sign diagram: $\dfrac{x^2-1}{2x+4} \Rightarrow \dfrac{(+)}{(-)} = -$. So $\boxed{\displaystyle\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x^2-1}{2x+4} = -\infty}$. So the graph is:



□

Exercise 2.6.80

Exercise 2.6.80. Find a function g that satisfies the conditions $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$. Graph $y = g(x)$ in a way that reflects the asymptotic behavior.

Solution. Since we want $\lim_{x\to\pm\infty} g(x) = 0$, then the graph of $y = g(x)$ will have $y = 0$ as a horizontal asymptote. Since we want $\lim_{x\to 3^-} g(x) = -\infty$ and $\lim_{x\to 3^+} g(x) = \infty$, then the graph of $y = g(x)$ has a vertical asymptote of $x = 3$.

Exercise 2.6.80

Exercise 2.6.80. Find a function g that satisfies the conditions $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$. Graph $y = g(x)$ in a way that reflects the asymptotic behavior.

Solution. Since we want $\lim_{x\to\pm\infty} g(x) = 0$, then the graph of $y = g(x)$ will have $y = 0$ as a horizontal asymptote. Since we want $\lim_{x\to 3^-} g(x) = -\infty$ and $\lim_{x\to 3^+} g(x) = \infty$, then the graph of $y = g(x)$ has a vertical asymptote of $x = 3$. We try to find a rational function, $g(x) = p(x)/q(x)$, satisfying these conditions. If we make polynomial p of degree less than that of polynomial q, then this will give (as we will check) the horizontal asymptote $y = 0$.

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Exercise 2.6.80 (continued 1)

Solution (continued). We have

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} \frac{1}{x-3} = \lim_{x\to\pm\infty} \frac{1}{x-3}\left(\frac{1/x}{1/x}\right)$$ dividing the numerator and denominator by the effective highest power of x in the denominator

$$= \lim_{x\to\pm\infty} \frac{1/x}{(x-3)/x} = \lim_{x\to\pm\infty} \frac{1/x}{1-3/x}$$ since $x \to \pm\infty$ then we can assume that $x \neq 0$

$$= \frac{\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 1 - 3\lim_{x\to\pm\infty} 1/x}$$ by the Difference, Constant Mult., and Quotient Rules, Theorem 2.12(2,4,5)

$$= \frac{(0)}{1-3(0)} = \frac{0}{1} = 0$$ by Example 2.6.1.

So $y = 0$ and a horizontal asymptote of the graph of $y = g(x)$, as desired.

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$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} \frac{1}{x-3} = \lim_{x\to\pm\infty} \frac{1}{x-3}\left(\frac{1/x}{1/x}\right) \text{ dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}$$

$$= \lim_{x\to\pm\infty} \frac{1/x}{(x-3)/x} = \lim_{x\to\pm\infty} \frac{1/x}{1-3/x} \text{ since } x \to \pm\infty \text{ then we can assume that } x \neq 0$$

$$= \frac{\lim_{x\to\pm\infty} 1/x}{\lim_{x\to\pm\infty} 1 - 3\lim_{x\to\pm\infty} 1/x} \text{ by the Difference, Constant Mult., and Quotient Rules, Theorem 2.12(2,4,5)}$$

$$= \frac{(0)}{1-3(0)} = \frac{0}{1} = 0 \text{ by Example 2.6.1.}$$

So $y = 0$ and a horizontal asymptote of the graph of $y = g(x)$, as desired.

Exercise 2.6.80 (continued 2)

Solution (continued). Since $\lim_{x\to 3} 1 = 1 \neq 0$ and $\lim_{x\to 3} x - 3 = 0$ (both by Theorem 2.2, say), then by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\lim_{x\to 3^\pm} g(x) = \pm\infty$. We consider one-sided limits (as required by the question).

For $\lim_{x\to 3^+} g(x)$, we analyze the sign of $\dfrac{1}{x-3}$ for "appropriate" x (since $x \to 3^+$, then appropriate x are close to 3 and slightly greater than 3). For such x, we have 1 is positive and $x - 3$ is positive (since x is greater than 3; so $x - 3$ is positive and "close to" 0). Combining the factors we get the sign diagram: $\dfrac{1}{x-3} \Rightarrow \dfrac{(+)}{(+)} = +$. So $\boxed{\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} \dfrac{1}{x-3} = \infty}$, as desired.

Exercise 2.6.80 (continued 2)

Solution (continued). Since $\lim_{x\to 3} 1 = 1 \neq 0$ and $\lim_{x\to 3} x - 3 = 0$ (both by Theorem 2.2, say), then by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\lim_{x\to 3^\pm} g(x) = \pm\infty$. We consider one-sided limits (as required by the question).

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For $\lim_{x\to 3^-} g(x)$, we analyze the sign of $\dfrac{1}{x-3}$ for "appropriate" x (since $x \to 3^-$, then appropriate x are close to 3 and slightly less than 3). For such x, we have 1 is positive and $x - 3$ is negative (since x is less than 3; so $x - 3$ is negative and "close to" 0).

Exercise 2.6.80 (continued 2)

Solution (continued). Since $\lim_{x\to 3} 1 = 1 \neq 0$ and $\lim_{x\to 3} x - 3 = 0$ (both by Theorem 2.2, say), then by Dr. Bob's Infinite Limits Theorem (applied to rational functions) $\lim_{x\to 3^\pm} g(x) = \pm\infty$. We consider one-sided limits (as required by the question).

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For $\lim_{x\to 3^-} g(x)$, we analyze the sign of $\dfrac{1}{x-3}$ for "appropriate" x (since $x \to 3^-$, then appropriate x are close to 3 and slightly less than 3). For such x, we have 1 is positive and $x - 3$ is negative (since x is less than 3; so $x - 3$ is negative and "close to" 0).

Exercise 2.6.80 (continued 3)

Exercise 2.6.80. Find a function g that satisfies the conditions $\lim_{x\to\pm\infty} g(x) = 0$, $\lim_{x\to 3^-} g(x) = -\infty$, and $\lim_{x\to 3^+} g(x) = \infty$. Graph $y = g(x)$ in a way that reflects the asymptotic behavior.

Solution (continued). Combining the factors we get the sign diagram: $\dfrac{1}{x-3} \Rightarrow \dfrac{(+)}{(-)} = -$. So

$$\boxed{\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} \frac{1}{x-3} = -\infty},$$

as desired.

We then have the graph:

Exercise 2.6.80 (continued 3)

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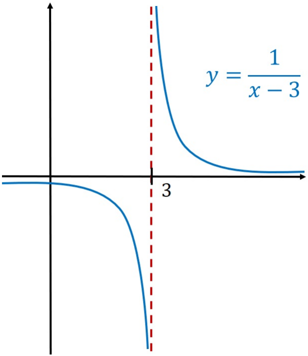

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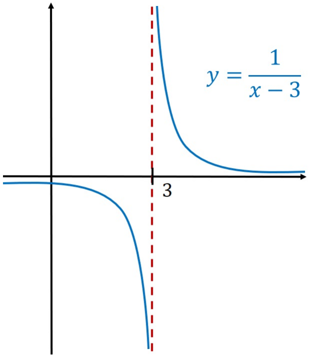

□

Exercise 2.6.102

Exercise 2.6.102. Use the formal definition of an infinite one-sided limit to prove that $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$.

Proof. Let f be a function defined on an interval (a, c), where $a < c$. We say that $f(x)$ *approaches negative infinity as* x *approaches* c *from the left*, and we write $\lim_{x \to c^-} f(x) = -\infty$, if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$c - \delta < x < c \text{ implies } f(x) < -B.$$

(This is the solution to Exercise 2.6.99(c).)

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(This is the solution to Exercise 2.6.99(c).) Notice that $f(x) = 1/(x-2)$ is defined on the interval $(-\infty, 2)$ (here, $c = 2$). Let $-B$ be any negative real number. Choose $\delta = 1/B > 0$.

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(This is the solution to Exercise 2.6.99(c).) Notice that $f(x) = 1/(x-2)$ is defined on the interval $(-\infty, 2)$ (here, $c = 2$). Let $-B$ be any negative real number. Choose $\delta = 1/B > 0$. If $2 - \delta < x < 2$ then $-\delta < x - 2 < 0$ and $-1/\delta > 1/(x-2)$, since $1/x$ is a decreasing function for negative input values. Now $\delta = 1/B$ so $1/\delta = B$ and $-1/\delta = -B$. So $2 - \delta < x < 2$ implies $f(x) = 1/(x-2) < -B$. Therefore, by the definition above, $\lim_{x \to 2^-} 1/(x-2) = -\infty$. □

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