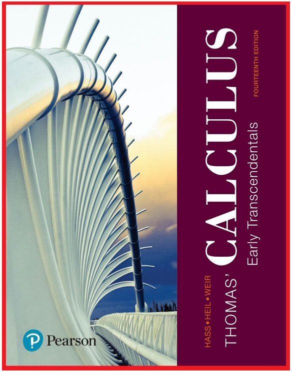
Calculus 1

Chapter 3. Derivatives

3.10. Related Rates—Examples and Proofs



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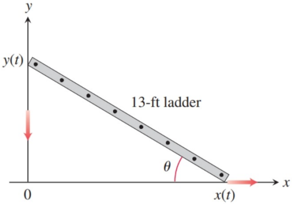
A 13 ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. **(a)** How fast is the top of the ladder sliding down the wall then? **(b)** At what rate is the area of the triangle formed by the ladder, wall, and ground changing then? **(c)** At what rate is the angle θ between the ladder and the ground changing then?



Solution. First, we have that the distance from the top of the ladder to the ground, y, and the distance from the bottom of the ladder to the wall, x, are both functions of time: $x = x(t)$ and $y = y(t)$. The rate at which the base of the ladder is moving is dx/dt and the rate at which the top of the ladder is moving is dy/dt.

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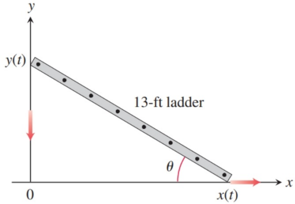

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(a) The question in terms of these symbols is: $dy/dt = ?$ when $x = 12$ ft and $dx/dt = 5$ ft/sec.

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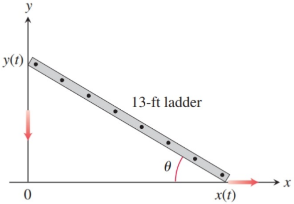

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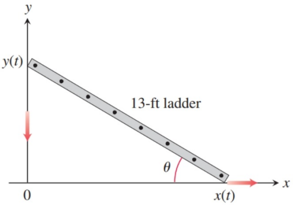

Solution (continued). Next, since the ladder forms a right triangle with sides of lengths x and y and hypotenuse of length 13 ft, then we have the relationship between the variables of $x^2 + y^2 = 13^2 = 169$, by the Pythagorean Theorem.

We differentiate this relationship with respect to time t to find a relationship between the rates: $\dfrac{d}{dt}[x^2 + y^2] = \dfrac{d}{dt}[169]$ or

$2x[dx/dt] + 2y[dy/dt] = 0$ or (since the question involves dy/dt) $dy/dt = -(x/y)(dx/dt)$.

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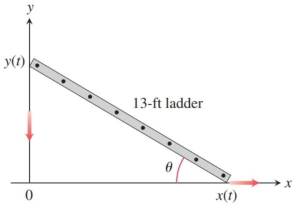

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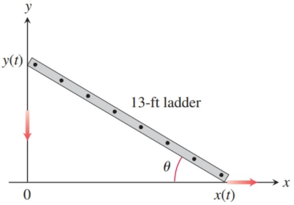

Solution (continued). The question is $dy/dt = ?$ when $x = 12$ ft and $dx/dt = 5$ ft/sec, and we have $dy/dt = -(x/y)(dx/dt)$. We still need to find y when $x = 12$ ft. We have the relationship $x^2 + y^2 = 169$ from above, so when $x = 12$ ft then $(12)^2 + y^2 = 169$ or $y^2 = 169 - 144 = 25$, and we see that $y = 5$ ft (notice, for physical reasons, that $y \geq 0$, so we ignore the algebraic possibility that $y = -5$ ft).

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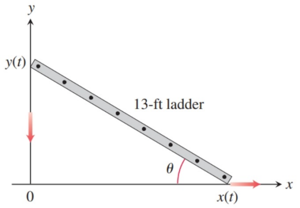

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$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{12 \text{ ft}}{5 \text{ ft}} 5 \text{ ft/sec} = \boxed{-12 \text{ ft/sec}}.$$

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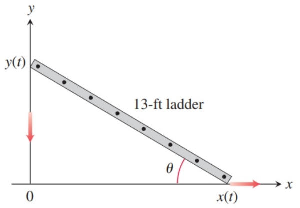

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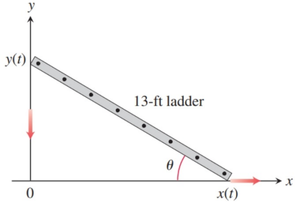
Exercise 3.10.23 (continued 3)

Exercise 3.10.23. A Sliding Ladder.
A 13 ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec. **(b)** At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?



Solution (continued). (b) From part (a) we know that when $x = 12$ ft and $dx/dt = 5$ ft/sec, then $y = 5$ ft and $dy/dt = -12$ ft/sec.

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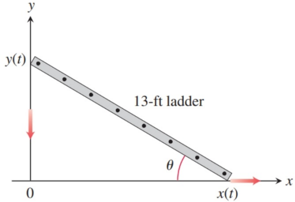

Solution (continued). (b) From part (a) we know that when $x = 12$ ft and $dx/dt = 5$ ft/sec, then $y = 5$ ft and $dy/dt = -12$ ft/sec.

With A as the area of the triangle, the question is $dA/dt = ?$ when $x = 12$ ft, $dx/dt = 5$ ft/sec, $y = 5$ ft, and $dy/dt = -12$ ft/sec.

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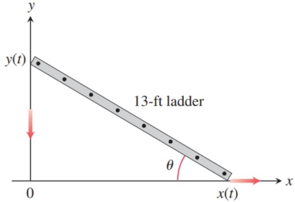

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The area of the triangle is 1/2 the base times the height of the triangle so, since we have a right triangle with base x and height y, the area is $A = xy/2$.

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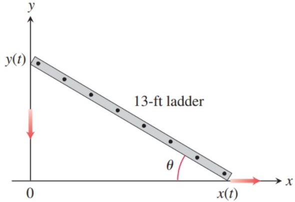

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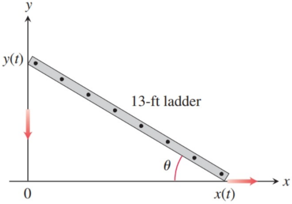

Solution (continued). We differentiate this relationship with respect to time t to find a relationship between the rates: $\frac{d}{dt}[A] = \frac{d}{dt}\left[\frac{xy}{2}\right]$ or $\frac{dA}{dt} = \frac{1}{2}\left(\left[\frac{dx}{dt}\right](y) + (x)\left[\frac{dy}{dt}\right]\right)$ or $dA/dt = (y\,dx/dt + x\,dy/dt)/2$.

We then have at the desired point in time:

$$dA/dt = ((5\text{ ft})(5\text{ ft/sec}) + (12\text{ ft})(-12\text{ ft/sec}))/2$$

$$= (25 - 144)/2\text{ ft}^2/\text{sec} = \boxed{-119/2\text{ ft}^2/\text{sec}}. \qquad \square$$

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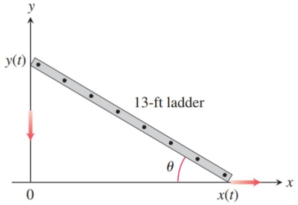

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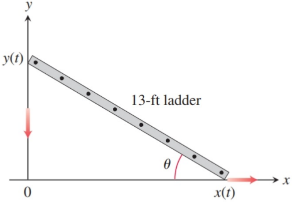

Solution (continued). (c) From part (a) we know that when $x = 12$ ft and $dx/dt = 5$ ft/sec, then $y = 5$ ft and $dy/dt = -12$ ft/sec.

With θ as the angle, the question is $d\theta/dt =?$ when $x = 12$ ft, $dx/dt = 5$ ft/sec, $y = 5$ ft, and $dy/dt = -12$ ft/sec.

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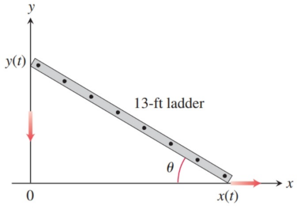

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We have from the right triangle that $\tan\theta = y/x$.

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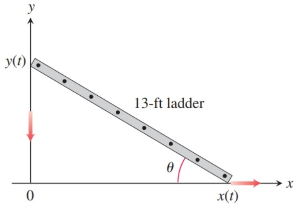

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Exercise 3.10.23 (continued 6)

Solution (continued). We differentiate this relationship with respect to time t to find a relationship between the rates: $\frac{d}{dt}[\tan\theta] = \frac{d}{dt}\left[\frac{y}{x}\right]$ or

$$\sec^2\theta \left[\frac{d\theta}{dt}\right] = \frac{[dy/dt](x) - (y)[dx/dt]}{(x)^2} \text{ or}$$

$$\frac{d\theta}{dt} = \frac{[dy/dt](x) - (y)[dx/dt]}{(x)^2}\cos^2\theta.$$

From part (a) we know that when $x = 12$ ft and $dx/dt = 5$ ft/sec, then $y = 5$ ft and $dy/dt = -12$ ft/sec. Notice that at this point in time, $\cos\theta = 12/13$.

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From part (a) we know that when $x = 12$ ft and $dx/dt = 5$ ft/sec, then $y = 5$ ft and $dy/dt = -12$ ft/sec. Notice that at this point in time, $\cos\theta = 12/13$.

We then have at the desired point in time:

$$d\theta/dt = \frac{(-12\text{ ft/sec})(12\text{ ft}) - (5\text{ ft})(5\text{ ft/sec})}{(12\text{ ft})^2}\left(\frac{12}{13}\right)^2$$

$$= \frac{-144-25}{144}\frac{144}{169}/\text{sec} = \boxed{-1/\text{sec}}. \qquad \square$$

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Exercise 3.10.28

Exercise 3.10.28. A Draining Conical Reservoir.
Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
(a) How fast (centimeters per minute) is the water level falling when the water is 5 m deep? **(b)** How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

Solution. **(a)** (1) We draw a picture of the reservoir both in perspective and in cross section, where we let r and h be the radius and height/depth, respectively, of the cone of water and (not to scale):

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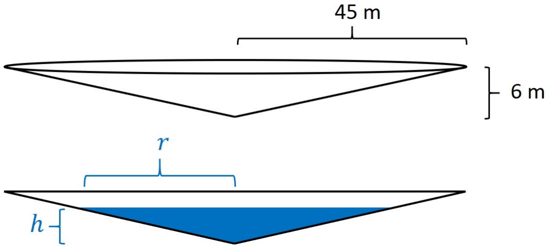

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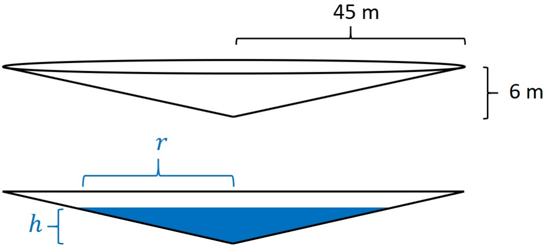

Exercise 3.10.28 (continued 1)

Solution (continued). (2) We let the volume of the cone of water be V so that $dV/dt = -50$ m^3/min. We are interested in the point in time when $h = 5$ m.

(3) The question is $dh/dt = ?$ when $dV/dt = -50$ m^3/min and $h = 5$ m.

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(3) The question is $dh/dt =$? when $dV/dt = -50$ m^3/min and $h = 5$ m.

(4) We need to relate V to the other variables h and r. Since the water forms a cone, then $V = \frac{1}{3}\pi r^2 h$. By similar triangles, we have $r/h = 45/6 = 15/2$ or $r = 15h/2$:

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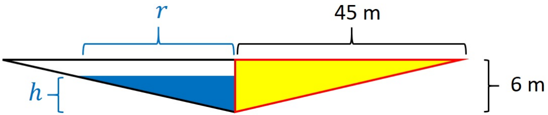

Hence we have $V = \frac{1}{3}\pi \left(\frac{15h}{2}\right)^2 h = \frac{75\pi h^3}{4}$.

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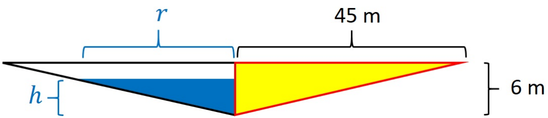
(4) We need to relate V to the other variables h and r. Since the water forms a cone, then $V = \frac{1}{3}\pi r^2 h$. By similar triangles, we have $r/h = 45/6 = 15/2$ or $r = 15h/2$:



Hence we have $V = \dfrac{1}{3}\pi\left(\dfrac{15h}{2}\right)^2 h = \dfrac{75\pi h^3}{4}$.

Exercise 3.10.28 (continued 2)

Solution (continued). (5) Differentiation of $V = \dfrac{75\pi h^3}{4}$ with respect to time gives $\dfrac{dV}{dt} = \dfrac{75\pi[3h^2]}{4}\left[\dfrac{dh}{dt}\right]$ or $\dfrac{dh}{dt} = \dfrac{4}{225\pi h^2}\dfrac{dV}{dt}$.

(6) When $dV/dt = -50$ m^3/min and $h = 5$ m, we have

$$\frac{dh}{dt} = \frac{4}{225\pi(5)^2}(-50)\text{ m/min} = \frac{-8}{225\pi}\text{ m/min}.$$

In units of cm/min we have

$$\frac{dh}{dt} = \frac{-8}{225\pi}\text{ m/min}\left(\frac{100\text{ cm}}{1\text{ m}}\right) = \frac{-32}{9\pi}\text{ cm/min} \approx -1.13\text{ cm/min},$$

so the water level is *changing* at a rate $\boxed{-32/(9\pi)\text{ cm/min}}$. □

Exercise 3.10.28 (continued 2)

Solution (continued). (5) Differentiation of $V = \dfrac{75\pi h^3}{4}$ with respect to time gives $\dfrac{dV}{dt} = \dfrac{75\pi[3h^2]}{4}\left[\dfrac{dh}{dt}\right]$ or $\dfrac{dh}{dt} = \dfrac{4}{225\pi h^2}\dfrac{dV}{dt}$.

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Exercise 3.10.28 (continued 3)

Exercise 3.10.28. A Draining Conical Reservoir.
Water is flowing at the rate of 50 $\mathrm{m}^3/\mathrm{min}$ from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
(b) How fast is the radius of the water's surface changing then? Answer in centimeters per minute.
Solution (continued). (1) We have the picture in part (a).

(2) We are interested in the point in time when $h = 5$ m.

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(3) The question is $dr/dt = ?$ when $dV/dt = -50$ m^3/min and $h = 5$ m. We saw in part (a) that $\frac{dh}{dt} = \frac{-32}{9\pi}$ cm/min at this point in time.

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(4) We need to relate r to the other variables. We saw in part (a) that by similar triangles, $r = 15h/2$.

Exercise 3.10.28 (continued 3)

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(5) Differentiation of $r = 15h/2$ with respect to time gives $\dfrac{dr}{dt} = \dfrac{15}{2}\dfrac{dh}{dt}$.

Exercise 3.10.28 (continued 3)

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(2) We are interested in the point in time when $h = 5$ m.

(3) The question is $dr/dt =$? when $dV/dt = -50 \text{ m}^3/\text{min}$ and $h = 5$ m. We saw in part (a) that $\dfrac{dh}{dt} = \dfrac{-32}{9\pi}$ cm/min at this point in time.

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(5) Differentiation of $r = 15h/2$ with respect to time gives $\dfrac{dr}{dt} = \dfrac{15}{2}\dfrac{dh}{dt}$.

Exercise 3.10.28 (continued 4)

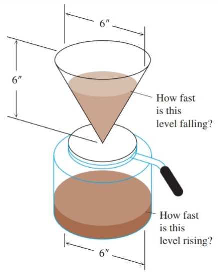
Exercise 3.10.28. A Draining Conical Reservoir.
Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.
(b) How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

Solution (continued). (6) When $dV/dt = -50 \text{ m}^3/\text{min}$ and $h = 5$ m, we have (by part (a)) that $\dfrac{dh}{dt} = \dfrac{-32}{9\pi}$ cm/min, and so

$$\frac{dr}{dt} = \frac{15}{2}\left(\frac{-32}{9\pi}\text{ cm/min}\right) = \boxed{-\frac{80}{3\pi}\text{ cm/min}} \approx -8.49 \text{ cm/min.} \ \square$$

Exercise 3.10.34

Exercise 3.10.34. Making Coffee.
Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.
(a) How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?
(b) How fast is the level in the cone falling then?



Solution. (1) We are given a picture, but we take a cross section of it and label constants and the variables (here the variables are the height of the coffee in the filter h_f, the radius of the top of the coffee in the filter r_f, and the height of the coffee in the pot h_p):

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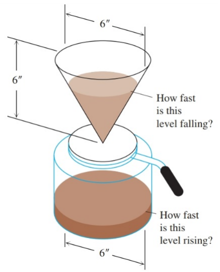

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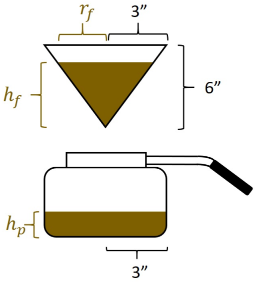

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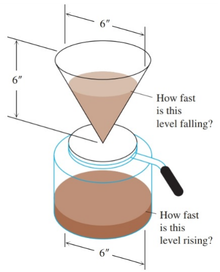

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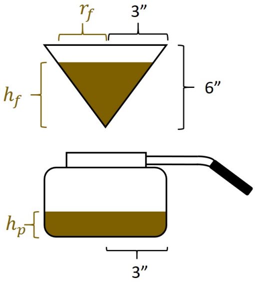

Exercise 3.10.34 (continued 1)

Exercise 3.10.34. Making Coffee.
Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. **(a)** How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

Solution (continued). (2) We let the volume of the coffee in the pot be V_p. We are interested in the point in time when $dV_p/dt = 10$ in^3/min and $h_f = 5$ in.

(3) The question is $dh_p/dt =?$ when $dV_p/dt = 10$ in^3/min and $h_f = 5$ in.

Exercise 3.10.34 (continued 1)

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(4) We need to relate h_p to the other variables. In the pot (idealized as a cylinder), we have the volume V_p as $V_p = \pi(3)^2 h_p = 9\pi h_p$ in^3 (since the volume of a cylinder of radius r and height h is $V = \pi r^2 h$).

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Exercise 3.10.34. Making Coffee.
Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. **(a)** How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

Solution (continued). (5) Differentiation of V_p with respect to time gives $dV_p/dt = 9\pi\, dh_p/dt$ or $\dfrac{dh_p}{dt} = \dfrac{1}{9\pi}\dfrac{dV_p}{dt}$.

(6) When $dV_p/dt = 10$ in^3/min and $h_f = 5$ in, we have $\dfrac{dh_p}{dt} = \dfrac{1}{9\pi}(10) = \boxed{\dfrac{10}{9\pi}\text{ in/min}} \approx 0.35$ in/min (notice that the answer here is independent of h_f). □

Exercise 3.10.34 (continued 2)

Exercise 3.10.34. Making Coffee.
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Exercise 3.10.34 (continued 3)

Exercise 3.10.34. Making Coffee.
Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. **(b)** How fast is the level in the cone falling then?

Solution (continued). (1) We have a picture above from part (a).

(2) We let the volume of the coffee in the filter be V_f. We are interested in the point in time when $-dV_f/dt = dV_p/dt = 10 \text{ in}^3/\text{min}$ and $h_f = 5$ in.

Exercise 3.10.34 (continued 3)

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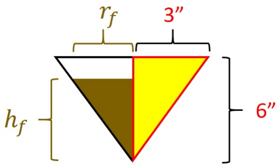
Solution (continued). (4) We need to relate h_f to the other variables. In the conical filter we have the volume of the coffee as $V_f = \frac{1}{3}\pi r_f^2 h_f$ (the volume of a cone of radius r and height h is $V = \frac{1}{3}\pi r^2 h$). Notice that by similar triangles, $r_f/h_f = 3/6$ or $r_f = h_f/2$:



So we have $V_f = \frac{1}{3}\pi r_f^2 h_f = \frac{1}{3}\pi \left(\frac{h_f}{2}\right)^2 h_f = \frac{1}{12}\pi h_f^3$.

(5) Differentiation of V_f with respect to time gives $dV_f/dt = \frac{1}{4}\pi h_f^2 dh_f/dt$ or $\dfrac{dh_f}{dt} = \dfrac{4dV_f/dt}{\pi h_f^2}$.

Exercise 3.10.34 (continued 4)

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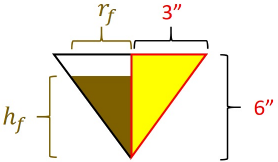

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(6) When $dV_f/dt = -dV_p/dt = -10$ in^3/min, $h_f = 5$ in we have
$$\frac{dh_f}{dt} = \frac{4(-10)}{\pi(5)^2} = \boxed{\frac{-8}{5\pi}\text{ in/min}} \approx -0.51 \text{ in/min.} \quad \square$$

Exercise 3.10.34 (continued 4)

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Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min. **(b)** How fast is the level in the cone falling then?

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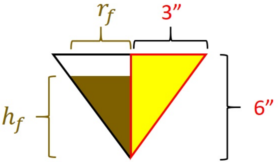

So we have $V_f = \frac{1}{3}\pi r_f^2 h_f = \frac{1}{3}\pi \left(\frac{h_f}{2}\right)^2 h_f = \frac{1}{12}\pi h_f^3$.
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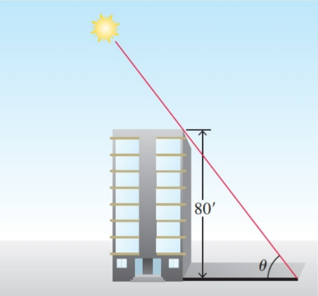
Theorem 3.10.40

Theorem 3.10.40. A Building's Shadow. On a morning of a day when the sun will pass directly overhead, the shadow of an 80 ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27°/\text{min}$. At what rate is the shadow decreasing? Express your answer in inches per minute to the nearest tenth.



Solution. First, they are mixing together units; they have given a rate of change of an angle in degrees/min and we must express angles in terms of radians (for the usual differentiation formulas to hold), and they have given us a distance in feet and asked for an answer in inches.

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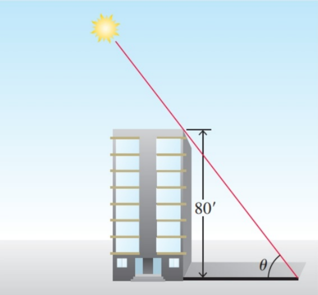

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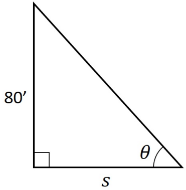
Theorem 3.10.40 (continued 1)

Solution (continued). (1) We draw a simplified picture and label the length of the shadow s:



(2) We are interested in the point in time when $s = 60$ ft and $d\theta/dt = 0.27°/\text{min} = (0.27°/\text{min})(\pi/180°) = 0.0015\pi/\text{min}$ (remember that radians are unitless).

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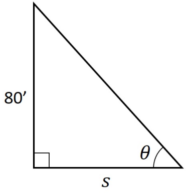

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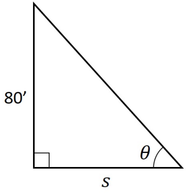
(3) The question is $ds/dt =?$ when $s = 60$ ft and $d\theta/dt = 0.0015\pi/\text{min}$.

Theorem 3.10.40 (continued 1)

Solution (continued). (1) We draw a simplified picture and label the length of the shadow s:



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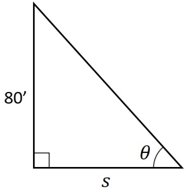
(3) The question is $ds/dt = ?$ when $s = 60$ ft and $d\theta/dt = 0.0015\pi/\text{min}$.

(4) We need to related s and θ. Notice that $\tan\theta = 80/s = 80s^{-1}$.

Theorem 3.10.40 (continued 1)

Solution (continued). (1) We draw a simplified picture and label the length of the shadow s:



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(4) We need to related s and θ. Notice that $\tan\theta = 80/s = 80s^{-1}$.

Theorem 3.10.40 (continued 2)

Solution (continued). (5) Differentiation with respect to t gives $\sec^2\theta[d\theta/dt] = 80[-s^{-2}[ds/dt]]$ or $\dfrac{ds}{dt} = \dfrac{-s^2\sec^2\theta}{80}\dfrac{d\theta}{dt}$.

(6) Notice that when $x = 60$ ft then, by the Pythagorean Theorem, the hypotenuse of the right triangle is $\sqrt{(60)^2+(80)^2} = 100$ ft, and so $\sec\theta = 100/60 = 5/3$ then. When $s = 60$ ft, $\sec\theta = 5/3$, and $d\theta/dt = 0.0015\pi/\text{min}$, we have

$$\frac{ds}{dt} = \frac{-(60)^2(5/3)^2}{80}0.0015\pi \text{ ft/min} = -0.1875\pi \text{ ft/min}.$$

Converting from feet to inches, we get $(-0.1875\pi \text{ ft/min})(12 \text{ in/ft}) = \boxed{-2.25\pi \text{ in/min}} \approx -7.07 \text{ in/min}$. □

Theorem 3.10.40 (continued 2)

Solution (continued). (5) Differentiation with respect to t gives $\sec^2 \overset{\curvearrowright}{\theta}[d\theta/dt] = 80[-s^{-\overset{\curvearrowright}{2}}[ds/dt]]$ or $\dfrac{ds}{dt} = \dfrac{-s^2 \sec^2 \theta}{80} \dfrac{d\theta}{dt}$.

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Exercise 3.10.44

Exercise 3.10.44. Ships.
Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

Solution. (1) We let a be the distance from O to A, b the distance from O to B, and c the distance from A to B. We then have the picture:

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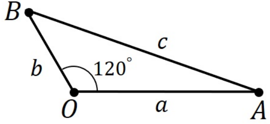

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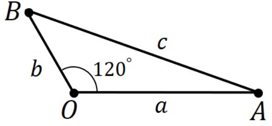

(2) We are given $da/dt = 14$ knots, $db/dt = 21$ knots, and we are interested in the point in time when $a = 5$ nautical miles and $b = 3$ nautical miles.

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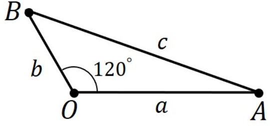

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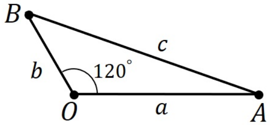
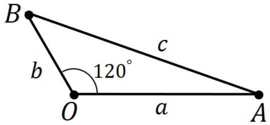
Exercise 3.10.44 (continued 1)



Solution (continued). (3) The question is $dc/dt = ?$ when $da/dt = 14$ knots, $db/dt = 21$ knots, $a = 5$ nautical miles, and $b = 3$ nautical miles.

(4) We use the Law of Cosines to relate a, b, and c and we have $c^2 = a^2 + b^2 - 2ab\cos(120°)$ or $c^2 = a^2 + b^2 - 2ab(-1/2)$ or $c^2 = a^2 + b^2 + ab$.

Exercise 3.10.44 (continued 1)



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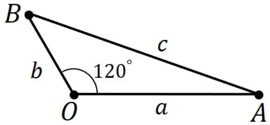
(5) Differentiating this relationship with respect to time t gives

$$2c\left[\frac{dc}{dt}\right] = 2a\left[\frac{da}{dt}\right] + 2b\left[\frac{db}{dt}\right] + \left(\left[\frac{da}{dt}\right](b) + (a)\left[\frac{db}{dt}\right]\right) \text{ or}$$

$$\frac{dc}{dt} = \frac{a}{c}\frac{da}{dt} + \frac{b}{c}\frac{db}{dt} + \frac{b}{2c}\frac{da}{dt} + \frac{a}{2c}\frac{db}{dt}.$$

Exercise 3.10.44 (continued 1)



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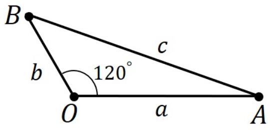
Exercise 3.10.44 (continued 2)



Solution (continued). (6) When $a = 5$ nautical miles and $b = 3$ nautical miles, then by the formula from (4) we have
$c^2 = (5)^2 + (3)^2 - 2(5)(3)(-1/2) = 25 + 9 + 15 = 49$, and so $c = 7$ nautical miles (c is a distance so $c \geq 0$). So at the desired point in time, we have

$$\frac{dc}{dt} = \frac{a}{c}\frac{da}{dt} + \frac{b}{c}\frac{db}{dt} + \frac{b}{2c}\frac{da}{dt} + \frac{a}{2c}\frac{db}{dt}$$

$$= \frac{(5)}{(7)}(14) + \frac{(3)}{(7)}(21) + \frac{(3)}{2(7)}(14) + \frac{(5)}{2(7)}(21) = \boxed{\frac{59}{2} \text{ knots}}.$$

Practice Exercise 3.148

Practice Exercise 3.148. Motion of a Particle.
A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

Solution. (1) For a picture, we consider the graph of $y = x^{3/2}$:

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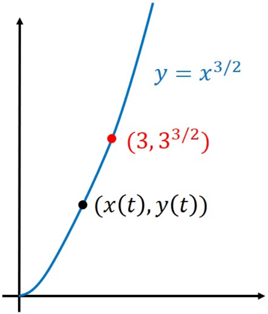

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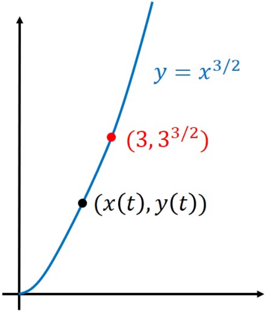

(2) We have points of the form $(x, x^{3/2})$ on the curve. Let D be the distance of the point to the origin.

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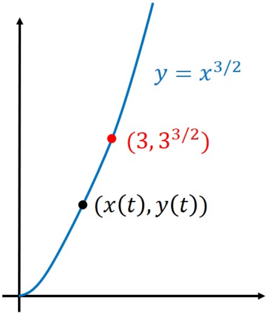

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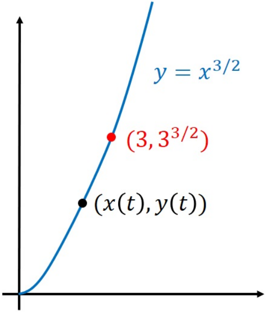

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Practice Exercise 3.148 (continued)

Solution (continued). (4) We need to relate the variables x, y, and D. By the Pythagorean Theorem, $D^2 = x^2 + y^2$.

(5) Differentiating this relationship with respect to time t gives $2\overset{\curvearrowright}{D}[dD/dt] = 2\overset{\curvearrowright}{x}[dx/dt] + 2\overset{\curvearrowright}{y}[dy/dt]$. Since $y = x^{3/2}$ then $\dfrac{dy}{dt} = \dfrac{3}{2}x^{1/2}\dfrac{dx}{dt}$, and so $2D\dfrac{dD}{dt} = 2x\dfrac{dx}{dt} + 2y\left(\dfrac{3}{2}x^{1/2}\dfrac{dx}{dt}\right)$. So $2D\dfrac{dD}{dt} = \dfrac{dx}{dt}\left(2x + 3x^{1/2}y\right)$ or

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(6) When $x = 3$ units then $y = 3^{3/2}$ and $D = \sqrt{(3)^2 + (3^{3/2})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$ units. So when $x = 3$ units and $dD/dt = 11$ units/sec, we have

$$\frac{dx}{dt} = \frac{2(6)}{2(3) + 3(3)^2}(11) = \frac{(12)(11)}{33} = \boxed{4} \text{ units/sec.} \quad \square$$

Practice Exercise 3.148 (continued)

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