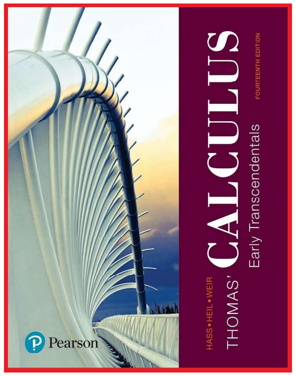
Calculus 1

Chapter 3. Derivatives

3.11. Linearization and Differentials—Examples and Proofs



Exercise 3.11.2

Exercise 3.11.2. Find the linearization $L(x)$ of $f(x) = \sqrt{x^2+9}$ at $x = a = -4$.

Solution. We write $f(x) = (x^2+9)^{1/2}$ so that by the Chain Rule (Theorem 3.2) we have $f'(x) = (1/2)(x^2+9)^{-1/2}[2x] = x/\sqrt{x^2+9}$. Now $f(a) = f(-4) = \sqrt{(-4)^2+9} = \sqrt{25} = 5$ and $f'(a) = f'(-4) = (-4)/\sqrt{(-4)^2+9} = -4/5$, so

$$L(x) = f(a) + f'(a)(x-a) = f(-4) + f'(-4)(x-(-4))$$

$$= 5 + (-4/5)(x+4) = \boxed{(-4/5)x - 9/5}.$$

□

Exercise 3.11.28

Exercise 3.11.28. Find dy when $y = \sec(x^2-1)$.

Solution. With $f(x) = \sec(x^2-1)$ we have by the Chain Rule (Theorem 3.2) that

$$dy = f'(x)\,dx = \sec(x^2-1)\tan(x^2-1)[2x]\,dx$$

$$= \boxed{2x\sec(x^2-1)\tan(x^2-1)\,dx}.$$

□

Exercise 3.11.38

Exercise 3.11.38. Find dy when $y = e^{\tan^{-1}\sqrt{x^2+1}}$.

Solution. With $f(x) = e^{\tan^{-1}\sqrt{x^2+1}}$ we have by the Chain Rule (Theorem 3.2) that

$$dy = f'(x)\,dx = e^{\tan^{-1}\sqrt{x^2+1}}\left[\frac{1}{1+(\sqrt{x^2+1})^2}[(1/2)(x^2+1)^{-1/2}[2x]]\right]dx$$

$$= \boxed{\frac{x}{(x^2+2)\sqrt{x^2+1}}e^{\tan^{-1}\sqrt{x^2+1}}\,dx}.$$

□

Example 3.11.A

Example 3.11.A. Use differentials to estimate the value of $\sin 31°$.

Solution. First, we have $31° = 30° + 1° = \pi/6 + \pi/180$ (radians). We take $f(x) = \sin x$ so that $f'(x) = \cos x$. With $a = \pi/6$ and $\Delta x = dx = \pi/180$, we have:

$$\sin 31° = \sin(\pi/6 + \pi/180) = f(a + \Delta x) = f(a) + \Delta y$$

$$\approx f(a) + dy = f(a) + f'(a)\,dx = \sin(\pi/6) + \cos(\pi/6)(\pi/180)$$

$$= (1/2) + (\sqrt{3}/2)(\pi/180) = 1/2 + \sqrt{3}\pi/360 \approx \boxed{0.515115}. \quad \square$$

Using a calculator, we have $\sin 31° \approx 0.515038$. So linearization gives an approximation that is accurate to three decimal places (but not to four decimal places).

Exercise 3.11.44

Exercise 3.11.44. For $f(x) = x^3 - 2x + 3$, $x_0 = 2$, and $dx = 0.1$, find: **(a)** the change $\Delta f = f(x_0 + dx) - f(x_0)$, **(b)** the value of the estimate $df = f'(x_0)\,dx$, and **(c)** the approximation error $|\Delta f - df|$.

Solution. First, $f'(x) = 3x^2 - 2$.

(a) We have

$$\Delta f = f(x_0 + dx) - f(x_0) = f(2 + 0.1) - f(2) = f(2.1) - f(2)$$

$$= ((2.1)^3 - 2(2.1) + 3) - ((2)^3 - 2(2) + 3)) = 8.061 - 7 = \boxed{1.061}. \quad \square$$

(b) Next, $df = f'(x_0)\,dx = f'(2)\,dx = (3(2)^2 - 2)(0.1) = \boxed{1}$. $\square$

(c) Finally, $|\Delta f - df| = |1.061 - 1| = \boxed{0.061}$. $\square$

Exercise 3.11.56

Exercise 3.11.56. The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's: **(a)** surface area? **(b)** volume?

Proof. The surface area of such a cube is $A = 6x^2$ and the volume of such a cube is $V = x^3$. The edge x is measured with an error of at most 0.5%, so the percentage change in the edge is $dx/x \times 100\% \leq 0.5\%$ or $dx \leq 0.005x$.

(a) Since $A = 6x^2$ then $dA = 12x\,dx$ and the percentage change in area is

$$\frac{dA}{A}\times 100\% = \frac{12x\,dx}{6x^2}\times 100\% \leq \frac{12x(0.005x)}{6x^2}\times 100\% = 0.010\times 100\% = \boxed{1\%}.$$

(b) Since $V = x^3$ then $dV = 3x^2\,dx$ and the percentage change in volume is

$$\frac{dV}{V}\times 100\% = \frac{3x^2\,dx}{x^3}\times 100\% \leq \frac{3x^2(0.005x)}{x^3}\times 100\% = 0.015\times 100\% = \boxed{1.5\%}.$$

$\square$

Exercise 3.11.58

Exercise 3.11.58. Tolerance (a) About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value? **(b)** About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

Solution. The volume of a cylinder of diameter $D = 2r$ and height h is $V = \pi r^2 h$. So here,

$$V = \pi(D/2)^2(10) = 5\pi D^2/2 \text{ m}^3 \text{ and } dV = 5\pi[2D]/2\,dD = 5\pi D\,dD \text{ m}^3.$$

The surface area of the side of a cylinder of diameter $D = 2r$ and height h is

$$A = 2\pi rh = 2\pi(D/2)(10) = 10\pi D \text{ m}^2 \text{ and } dA = 10\pi dD \text{ m}^2.$$

Exercise 3.11.58 (continued)

Solution (continued). (a) We want $dV/V \times 100\% = 1\%$, so we require the percentage change in volume to satisfy

$$\frac{5\pi D\, dD}{5\pi D^2/2} \times 100\% = \frac{2dD}{D} \times 100\% = 1\%,$$

from which we need $dD/D \times 100\% = (1/2)\%$. That is, we need D to be measured with an accuracy of $(1/2)\% = \boxed{0.5\%}$.

(b) We want $dA/A \times 100\% = 5\%$, so we require the percentage change in surface to satisfy

$$\frac{10\pi dD}{10\pi D} \times 100\% = \frac{dD}{D} \times 100\% = 5\%,$$

from which we need $dD/D \times 100\% = 5\%$. That is, we need D to be measured with an accuracy of $\boxed{5\%}$. □

Lemma 3.11.A

Lemma 3.11.A. If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the corresponding change Δy in f is given by $\Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x$ in which $\varepsilon \to 0$ as $\Delta x \to 0$.

Proof. The approximation error $\Delta f - df$ at $x = a$ is

$$\Delta f - df = \Delta f - f'(a)\,dx = \Delta f - f'(a)\Delta x = (f(a+\Delta x) - f(a)) - f'(a)\Delta x$$

$$= \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)\Delta x = \varepsilon\,\Delta x \quad (*)$$

where $\varepsilon = \left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)$. Since $f'(a)$ exists by hypothesis, then as $\Delta x \to 0$ the difference quotient $\frac{f(a+\Delta x) - f(a)}{\Delta x}$ approaches $f'(a)$, so that $\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) = \varepsilon \to 0$ as $\Delta x \to 0$. Also, $\Delta y = \Delta f = f'(a)\Delta x + \varepsilon\,\Delta x$ from $(*)$, as claimed. □

Theorem 3.2

Theorem 3.2. The Chain Rule.
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and $(f \circ g)'(x) = f'(g(x))[g'(x)]$.

Proof. Let x_0 be a point at which g is differentiable and suppose f is differentiable at $g(x_0)$. We show that $\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0))g'(x_0)$ so that the claim then follows since x_0 is an arbitrary point satisfying the hypotheses.

Let Δx be an increment in x and let $\Delta u = g(x_0) - g(x_0 + \Delta x)$ and $\Delta y = f(u_0) - f(u_0 + \Delta u)$ be the corresponding increments in u and y. By Lemma 3.11.A, we have

$$\Delta u = g'(x_0)\,\Delta x + \varepsilon_1\,\Delta x = (g'(x_0) + \varepsilon_1)\,\Delta x,$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$.

Theorem 3.2 (continued 1)

Proof (continued). Similarly, with $u_0 = g(x_0)$,

$$\Delta y = f'(u_0)\,\Delta u + \varepsilon_2\,\Delta u = (f'(u_0) + \varepsilon_2)\,\Delta u,$$

where $\varepsilon_2 \to 0$ as $\Delta u \to 0$. Since g is differentiable at x_0 by hypothesis, then g is continuous at x_0 by Theorem 3.1 (Differentiability Implies Continuity) so $\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0}(g(x_0) - g(x_0 + \Delta x)) = 0$ and hence $\Delta u \to 0$ as $\Delta x \to 0$. We therefore have

$$\Delta y = (f'(u_0) + \varepsilon_2)\,\Delta u = (f'(u_0) + \varepsilon_2)(g'(x_0) + \varepsilon_1)\,\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \varepsilon_2 g'(x_0) + f'(u_0)\varepsilon_1 + \varepsilon_1\varepsilon_2.$$

As $\Delta x \to 0$ we have both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Theorem 3.2 (continued 2)

Theorem 3.2. The Chain Rule.
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and $(f \circ g)'(x) = f'(g(x))[g'(x)]$.

Proof (continued). As $\Delta x \to 0$ we have both $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Since $g'(x_0)$ and $f'(u_0)$ are some fixed numbers, then

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(f'(u_0)g'(x_0) + \varepsilon_2 g'(x_0) + f'(u_0)\varepsilon_1 + \varepsilon_1\varepsilon_2\right)$$

$$= f'(u_0)g'(x_0) + (0)g'(x_0) + f'(u_0)(0) + (0)(0)$$

$$= f'(u_0)g'(x_0) = f'(g(x_0))g'(x_0).$$

Since x_0 is an arbitrary point satisfying the hypotheses, then we have $dy/dx = f'(g(x))g'(x)$, as claimed. □