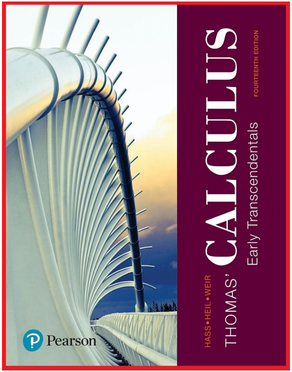
Calculus 1

Chapter 3. Derivatives

3.3. Differentiation Rules—Examples and Proofs



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Theorem 3.3.A

Theorem 3.3.A. Derivative of a Constant Function.
If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}[c] = 0.$$

Proof. From the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0.$$

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Theorem 3.3.B

Theorem 3.3.B. Power Rule for Positive Integers.
If n is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Proof. Notice that

$$(z-x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1})$$
$$= (z^n + z^{n-1}x + z^{n-2}x^2 + \cdots + z^3x^{n-3} + z^2x^{n-2} + zx^{n-1})$$
$$-(z^{n-1}x + z^{n-2}x^2 + z^{n-3}x^3 + \cdots + z^2x^{n-2} + zx^{n-1} + x^n) = z^n - x^n.$$

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$$(z-x)(z^{n-1}+z^{n-2}x+z^{n-3}x^2+\cdots+z^2x^{n-3}+zx^{n-2}+x^{n-1})$$
$$=(z^n+z^{n-1}x+z^{n-2}x^2+\cdots+z^3x^{n-3}+z^2x^{n-2}+zx^{n-1})$$
$$-(z^{n-1}x+z^{n-2}x^2+z^{n-3}x^3+\cdots+z^2x^{n-2}+zx^{n-1}+x^n)=z^n-x^n.$$

So by the alternative formula for the definition of the derivative (see Exercise 3.2.24) we have

$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{z\to x}\frac{z^n-x^n}{z-x}$$

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$$f'(x) = \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{z\to x}\frac{z^n-x^n}{z-x}$$

Theorem 3.3.B (continued)

Proof (continued).

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} = \lim_{z\to x} \frac{z^n - x^n}{z-x} \\ &= \lim_{z\to x} \frac{1}{z-x}(z-x)(z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots \\ &\quad + z^2x^{n-3} + zx^{n-2} + x^{n-1}) \\ &= \lim_{z\to x} (z^{n-1} + z^{n-2}x + z^{n-3}x^2 + \cdots + z^2x^{n-3} + zx^{n-2} + x^{n-1}) \\ &= (x)^{n-1} + (x)^{n-2}x + (x)^{n-3}x^2 + \cdots \\ &\quad + (x)^2x^{n-3} + (x)x^{n-2} + (x)^{n-1} \\ &= nx^{n-1}, \end{aligned}$$

as claimed. □

Theorem 3.3.D

Theorem 3.3.D. Derivative Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}.$$

Proof. Let $u(x)$ be differentiable and define $f(x) = cu(x)$. We want to show that $f'(x) = cu'(x)$. By the definition of derivative,

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} = \lim_{h\to 0}\frac{c(u(x+h)-u(x))}{h} \\ &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \text{ by the Constant Multiple} \\ &\quad \text{Rule for Limits, Theorem 2.1(3)} \\ &= cu'(x).\end{aligned}$$

That is, $\frac{d}{dx}[f(x)] = \frac{d}{dx}[cu(x)] = c\frac{d}{dx}[u(x)] = c\frac{du}{dx}$, as claimed. □

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&= c\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \text{ by the Constant Multiple} \\
&\quad \text{Rule for Limits, Theorem 2.1(3)} \\
&= cu'(x).
\end{aligned}$$

That is, $\frac{d}{dx}[f(x)] = \frac{d}{dx}[cu(x)] = c\frac{d}{dx}[u(x)] = c\frac{du}{dx}$, as claimed. □

Theorem 3.3.E

Theorem 3.3.E. Derivative Sum Rule
If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}[u+v] = \frac{du}{dx} + \frac{dv}{dx}.$$

Proof. Let $u(x)$ and $v(x)$ be differentiable and define $f(x) = u(x) + v(x)$. We want to show that $f'(x) = u'(x) + v'(x)$. By the definition of derivative,

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(u(x+h) + v(x+h)) - (u(x) + v(x))}{h} \\ &= \lim_{h \to 0} \frac{(u(x+h) - u(x)) + (v(x+h) - v(x))}{h} \end{aligned}$$

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Theorem 3.3.E (continued)

Theorem 3.3.E. Derivative Sum Rule
If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points, $\dfrac{d}{dx}[u+v] = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Proof (continued).

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(u(x+h)-u(x))+(v(x+h)-v(x))}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} + \lim_{h\to 0} \frac{v(x+h)-v(x)}{h} \\
&\quad \text{by the Sum Rule for Limits, Theorem 2.1(1)} \\
&= u'(x)+v'(x).
\end{aligned}$$

That is, $\dfrac{d}{dx}[f(x)] = \dfrac{d}{dx}[u(x)+v(x)] = \dfrac{d}{dx}[u(x)] + \dfrac{d}{dx}[v(x)]$, as claimed. □

Exercise 3.3.4(a)

Exercise 3.3.4(a). Find the derivative of $w = 3z^7 - 7z^3 + 21z^2$.

Solution. We have

$$\begin{aligned}
\frac{dw}{dz} &= \frac{d}{dz}[3z^7 - 7z^3 + 21z^2] = \frac{d}{dz}[3z^7] + \frac{d}{dz}[-7z^3] + \frac{d}{dz}[21z^2] \\
&\quad \text{by the Derivative Sum Rule, Theorem 3.3.E} \\
&= 3\frac{d}{dz}[z^7] - 7\frac{d}{dz}[z^3] + 21\frac{d}{dz}[z^2] \text{ by the Derivative Constant} \\
&\quad \text{Multiple Rule, Theorem 3.3.D} \\
&= 3[7z^{7-1}] - 7[3z^{3-1}] + 21[2z^{2-1}] \text{ by the Derivative Power Rule} \\
&\quad \text{for Positive Integers, Theorem 3.3.B} \\
&= \boxed{21z^6 - 21z^2 + 42z}. \qquad \square
\end{aligned}$$

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&\quad \text{by the Derivative Sum Rule, Theorem 3.3.E} \\
&= 3\frac{d}{dz}[z^7] - 7\frac{d}{dz}[z^3] + 21\frac{d}{dz}[z^2] \text{ by the Derivative Constant} \\
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Exercise 3.3.12(a)

Exercise 3.3.12(a). Find the derivative of $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$.

Solution. We have

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left[\frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}\right] = \frac{d}{d\theta}\left[\frac{12}{\theta}\right] + \frac{d}{d\theta}\left[-\frac{4}{\theta^3}\right] + \frac{d}{d\theta}\left[\frac{1}{\theta^4}\right]$$

by the Derivative Sum Rule, Theorem 3.3.E

$$= 12\frac{d}{d\theta}\left[\frac{1}{\theta}\right] - 4\frac{d}{d\theta}\left[\frac{1}{\theta^3}\right] + \frac{d}{d\theta}\left[\frac{1}{\theta^4}\right]$$ by the Derivative Constant Multiple Rule, Theorem 3.3.D

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$$= 12\frac{d}{d\theta}\left[\theta^{-1}\right] - 4\frac{d}{d\theta}\left[\theta^{-3}\right] + \frac{d}{d\theta}\left[\theta^{-4}\right]$$

$$= 12[-\theta^{-1-1}] - 4[-3\theta^{-3-1}] + [-4\theta^{-4-1}] = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5}$$

by the Derivative Power Rule (General Version), Theorem 3.3.C

$$= \boxed{\frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5}}. \qquad \square$$

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Exercise 3.3.34

Exercise 3.3.34. Find the derivative of $y = x^{-3/5} + \pi^{3/2}$.

Solution. We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[x^{-3/5} + \pi^{3/2}] = \frac{d}{dx}[x^{-3/5}] + \frac{d}{dx}[\pi^{3/2}] \\ &\quad \text{by the Derivative Sum Rule, Theorem 3.3.E} \\ &= [(-3/5)x^{-3/5-1}] + [0] \text{ by the Derivative Power Rule} \\ &\quad \text{(General Version), Theorem 3.3.C, and the Derivative of a} \\ &\quad \text{Constant Function, Theorem 3.3.A (since } \pi^{3/2} \text{ is constant)} \\ &= \boxed{\frac{-3}{5}x^{-8/5}}. \qquad \square \end{aligned}$$

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&\quad \text{by the Derivative Sum Rule, Theorem 3.3.E} \\
&= [(-3/5)x^{-3/5-1}] + [0] \text{ by the Derivative Power Rule} \\
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&\quad \text{Constant Function, Theorem 3.3.A (since } \pi^{3/2} \text{ is constant)} \\
&= \boxed{\frac{-3}{5}x^{-8/5}}. \qquad \square
\end{aligned}$$

Example 3.3.4

Example 3.3.4. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangent lines? If so, where?

Solution. First, we find the derivative:
$y' = \frac{d}{dx}[x^4 - 2x^2 + 2] = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$. Now a horizontal line has slope 0 and the slope of a line tangent to the graph of a curve $y = f(x)$ is $y' = f'(x)$, so we set y' equal to 0 and solve for x: $y' = 4x(x-1)(x+1) = 0$ implies $x = -1$, $x = 0$, or $x = 1$. So the curve has horizontal tangents at $\boxed{x = -1,\ x = 0,\text{ and } x = 1}$.

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Example 3.3.4 (continued)

Solution (continued). Since $y = f(x)$ is defined and differentiable on all of $\mathbb{R}$ then the graph of $y = f(x)$ is "smooth" (a term we will formalize in "Section 6.3. Arc Length"; "smooth" will then take on a slightly more involved meaning), the graph contains the points $(-1, 1)$, $(0, 2)$, and $(1, 1)$, and we can get a good idea of the graph of $y = f(x)$. The graph has horizontal tangents at the three mentioned points (and at no others), so the graph must look something like:

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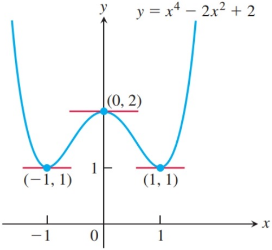

Figure 3.12

Example 3.3.4 (continued)

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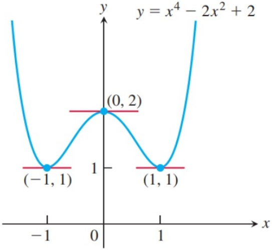

Figure 3.12

Example 3.3.A

Example 3.3.A. Differentiate $f(x) = x + 5e^x$.

Solution. We have

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x + 5e^x] = \frac{d}{dx}[x] + \frac{d}{dx}[5e^x] \text{ by the Derivative Sum Rule, Theorem 3.3.E} \\ &= \frac{d}{dx}[x] + 5\frac{d}{dx}[e^x] \text{ by the Derivative Constant Multiple Rule, Theorem 3.3.D} \\ &= [1] + 5[e^x] \text{ by Theorem 3.3.B and Theorem 3.3.F} \\ &= \boxed{1 + 5e^x}. \qquad \square \end{aligned}$$

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Theorem 3.3.G

Theorem 3.3.G. Derivative Product Rule
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = [u'](v) + (u)[v'].$$

Proof. Let $u(x)$ and $v(x)$ be differentiable and define $f(x) = u(x)v(x)$. We want to show that $f'(x) = u(x)v'(x) + u'(x)v(x)$.

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$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(u(x+h)v(x+h)) - (u(x)v(x))}{h} \\ &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \end{aligned}$$

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Proof. Let $u(x)$ and $v(x)$ be differentiable and define $f(x) = u(x)v(x)$. We want to show that $f'(x) = u(x)v'(x) + u'(x)v(x)$. By the definition of derivative,
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Theorem 3.3.G (continued 1)

Proof (continued). ...

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)(v(x+h) - v(x)) + v(x)(u(x+h) - u(x))}{h} \\
&= \lim_{h\to 0} \left(u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right) \\
&= \lim_{h\to 0} \left(u(x+h)\frac{v(x+h) - v(x)}{h} \right) + \lim_{h\to 0} \left(v(x)\frac{u(x+h) - u(x)}{h} \right) \\
&\quad \text{by the Sum Rule for Limits, Theorem 2.1(1)} \\
&= \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)[v'(x)] + [u'(x)]v(x),
\end{aligned}
$$

where $\lim_{h\to 0} u(x+h) = u(x)$ since u is continuous at x by Theorem 2.1.

Theorem 3.3.G (continued 1)

Proof (continued). ...

$$\begin{aligned}
&= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h)(v(x+h) - v(x)) + v(x)(u(x+h) - u(x))}{h} \\
&= \lim_{h\to 0} \left(u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h} \right) \\
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&= \lim_{h\to 0} u(x+h) \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} + v(x) \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\
&= u(x)[v'(x)] + [u'(x)]v(x),
\end{aligned}$$

where $\lim_{h\to 0} u(x+h) = u(x)$ since u is continuous at x by Theorem 2.1.

Theorem 3.3.G (continued 2)

Theorem 3.3.G. Derivative Product Rule
If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx} = [u'](v) + (u)[v'].$$

Proof (continued). ...

$$f'(x) \quad = \quad u(x)[v'(x)] + [u'(x)]v(x) = [u'(x)](v(x)) + (u(x))[v'(x)].$$

That is, $\frac{d}{dx}[f(x)] = \frac{d}{dx}[u(x)v(x)] = \frac{d}{dx}[u(x)](v(x)) + (u(x))\frac{d}{dx}[v(x)] = [u'](v) + (u)[v']$, as claimed. □

Example 3.3.B

Example 3.3.B. Differentiate $f(x) = (4x^3 - 5x^2 + 4)(7x^2 - x)$.

Solution. By the Derivative Product Rule, Theorem 3.3.G, we have

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(4x^3 - 5x^2 + 4)(7x^2 - x)] \\ &= \frac{d}{dx}[4x^3 - 5x^2 + 4](7x^2 - x) + (4x^3 - 5x^2 + 4)\frac{d}{dx}[7x^2 - x] \\ &= \boxed{[12x^2 - 10x](7x^2 - x) + (4x^3 - 5x^2 + 4)[14x - 1]}. \end{aligned}$$

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We can simplify this, but prefer to leave it in its current form. Notice that in expressing our answer we have put the parts of f which are differentiated (as required by the Derivative Product Rule) in square brackets, and the parts which are not differentiated (as required by the Derivative Product Rule) are in parentheses. □

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Theorem 3.3.H

Theorem 3.3.H. Derivative Quotient Rule.
If u and v are differentiable at x, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} = \frac{[u'](v) - (u)[v']}{(v)^2}.$$

Proof. Let $u(x)$ and $v(x)$ be differentiable and define $f(x) = u(x)/v(x)$. We want to show that $f'(x) = (u'(x)v(x) - u(x)v'(x))/(v(x))^2$.

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Proof. Let $u(x)$ and $v(x)$ be differentiable and define $f(x) = u(x)/v(x)$. We want to show that $f'(x) = (u'(x)v(x) - u(x)v'(x))/(v(x))^2$. By the definition of derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{v(x)u(x+h) - u(x)v(x+h)}{v(x+h)v(x)}\right) \end{aligned}$$

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Theorem 3.3.H (continued 1)

Proof (continued).

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)u(x+h)-u(x)v(x+h)}{v(x+h)v(x)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)u(x+h)-v(x)u(x)+v(x)u(x)-u(x)v(x+h)}{v(x+h)v(x)}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)(u(x+h)-u(x))-u(x)(v(x+h)-v(x))}{v(x+h)v(x)}\right) \\ &= \lim_{x\to 0}\frac{v(x)\frac{u(x+h)-u(x)}{h}-u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)} \\ &= \frac{\lim_{h\to 0} v(x)\frac{u(x+h)-u(x)}{h}-\lim_{h\to 0} u(x)\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0} v(x+h)v(x)} \end{aligned}$$

by the Quotient Rule for Limits, Theorem 2.1(5), assuming the denominator is not 0

Theorem 3.3.H (continued 1)

Proof (continued).

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)u(x+h)-u(x)v(x+h)}{v(x+h)v(x)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)u(x+h)-v(x)u(x)+v(x)u(x)-u(x)v(x+h)}{v(x+h)v(x)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{v(x)(u(x+h)-u(x))-u(x)(v(x+h)-v(x))}{v(x+h)v(x)}\right)\\
&= \lim_{x\to 0}\frac{v(x)\frac{u(x+h)-u(x)}{h}-u(x)\frac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}\\
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\end{aligned}$$

by the Quotient Rule for Limits, Theorem 2.1(5), assuming the denominator is not 0

Theorem 3.3.H (continued 2)

Proof (continued).

$$\begin{aligned} f'(x) &= \frac{\lim_{h\to 0} v(x)\frac{u(x+h)-u(x)}{h} - \lim_{h\to 0} u(x)\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0} v(x+h)v(x)} \\ &= \frac{v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}{v(x)\lim_{h\to 0} v(x+h)} \\ &\quad \text{by the Constant Multiple Rule for Limits, Theorem 2.1(3)} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{v^2(x)}, \end{aligned}$$

where $\lim_{h\to 0} v(x+h) = v(x)$ since v is continuous at x by Theorem 2.1.

That is, $\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] =$
$\frac{\left[\frac{d}{dx}[u(x)]\right](v(x)) - (u(x))\left[\frac{d}{dx}[v(x)]\right]}{v(x)^2} = \frac{[u'](v) - (u)[v']}{(v)^2}$, as claimed. □

Theorem 3.3.H (continued 2)

Proof (continued).

$$
\begin{aligned}
f'(x) &= \frac{\lim_{h\to 0} v(x)\frac{u(x+h)-u(x)}{h} - \lim_{h\to 0} u(x)\frac{v(x+h)-v(x)}{h}}{\lim_{h\to 0} v(x+h)v(x)} \\
&= \frac{v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} - u(x)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h}}{v(x)\lim_{h\to 0} v(x+h)}
\end{aligned}
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Exercise 3.3.20

Exercise 3.3.20. Differentiate $f(t) = \dfrac{t^2 - 1}{t^2 + t - 2}$.

Solution. By the Derivative Quotient Rule, Theorem 3.3.H, we have

$$\begin{aligned} f'(x) &= \frac{d}{dt}\left[\frac{t^2-1}{t^2+t-2}\right] \\ &= \frac{\frac{d}{dt}[t^2-1](t^2+t+2) - (t^2-1)\frac{d}{dt}[t^2+t-2]}{(t^2+t-2)^2} \\ &= \boxed{\frac{[2t](t^2+t+2) - (t^2-1)[2t+1]}{(t^2+t-2)^2}}. \end{aligned}$$

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We can simplify this, but prefer to leave it in its current form. Notice that in expressing our answer we have put the parts of f which are differentiated (as required by the Derivative Quotient Rule) in square brackets, and the parts which are not differentiated (as required by the Derivative Quotient Rule) are in parentheses. □

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Exercise 3.3.48

Example 3.3.48. Differentiate $u(x) = \dfrac{(x^2+x)(x^2-x+1)}{x^4}$.

Solution. We treat this as a quotient with a product in the numerator.

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Solution. We treat this as a quotient with a product in the numerator. We have

$$u'(x) = \frac{d}{dx}\left[\frac{(x^2+x)(x^2-x+1)}{x^4}\right]$$

$$= \frac{\frac{d}{dx}[(x^2+x)(x^2-x+1)]\,(x^4) - ((x^2+x)(x^2-x+1))\,\frac{d}{dx}\left[x^4\right]}{(x^2)^2}$$

$$= \boxed{\frac{[[2x+1](x^2-x+1)+(x^2+x)[2x-1]](x^4)-((x^2+x)(x^2-x+1))[4x^3]}{(x^2)^2}.}$$

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$$= \boxed{\frac{[[2x+1](x^2-x+1)+(x^2+x)[2x-1]](x^4) - ((x^2+x)(x^2-x+1))[4x^3]}{(x^2)^2}.}$$

We can simplify the result from here, but that will disguise the fact that we have used the Derivative Product Rule within the Derivative Quotient Rule. When we do applications later we will need to simplify derivatives, so for now we leave the answer as given. □

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We can simplify the result from here, but that will disguise the fact that we have used the Derivative Product Rule within the Derivative Quotient Rule. When we do applications later we will need to simplify derivatives, so for now we leave the answer as given. □

Exercise 3.3.78

Exercise 3.2.78. Power Rule for Negative Integers.

Prove that if m is a positive integer then $\frac{d}{dx}[x^{-m}] = -mx^{-m-1}$. HINT: Use the Derivative Quotient Rule (Theorem 3.3.H) and the Derivative Power Rule for Positive Integers (Theorem 3.3.B); notice that we have proved both of these already.

Proof. We can write x^{-m} as a quotient: $x^{-m} = \frac{1}{x^m}$.

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Proof. We can write x^{-m} as a quotient: $x^{-m} = \frac{1}{x^m}$. We then have:

$$\frac{d}{dx}[x^{-m}] = \frac{d}{dx}\left[\frac{1}{x^m}\right]$$

$$= \frac{\frac{d}{dx}[1](x^m) - (1)\frac{d}{dx}[x^m]}{(x^m)^2} \text{ by the Derivative Quotient Rule, Theorem 3.3.H}$$

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$$\begin{aligned} \frac{d}{dx}[x^{-m}] &= \frac{d}{dx}\left[\frac{1}{x^m}\right] \\ &= \frac{\frac{d}{dx}[1](x^m) - (1)\frac{d}{dx}[x^m]}{(x^m)^2} \text{ by the Derivative Quotient Rule,} \\ &\quad \text{Theorem 3.3.H} \end{aligned}$$

Exercise 3.3.78 (continued)

Exercise 3.2.78. Power Rule for Negative Integers.

Prove that if m is a positive integer then $\frac{d}{dx}[x^{-m}] = -mx^{-m-1}$. HINT: Use the Derivative Quotient Rule (Theorem 3.3.H) and the Derivative Power Rule for Positive Integers (Theorem 3.3.B); notice that have proved both of these already.

Proof (continued). ...

$$\begin{aligned} \frac{d}{dx}[x^{-m}] &= \frac{\frac{d}{dx}[1](x^m) - (1)\frac{d}{dx}[x^m]}{(x^m)^2} \\ &= \frac{[0](x^m) - (1)[(m)x^{m-1}]}{x^{2m}} \text{ by the Derivative Power Rule} \\ &\quad \text{for Positive Integers, Theorem 3.3.B} \\ &= \frac{-mx^{m-1}}{x^{2m}} = -mx^{(m-1)-2m} = -mx^{-m-1}, \end{aligned}$$

as claimed. □

Exercise 3.2.77(a,b)

Exercise 3.2.77(a,b). Generalizing the Product Rule.
The Derivative Product Rule tells us how to find the derivative of a product of *two* differentiable functions.
(a) Suppose functions $u_1(x)$, $u_2(x)$, and $u_3(x)$ are differentiable. Find the derivative of the product $u_1(x)u_2(x)u_3(x)$.
(b) Suppose functions $u_1(x)$, $u_2(x)$, $u_3(x)$, and $u_4(x)$ are differentiable. Find the derivative of the product $u_1(x)u_2(x)u_3(x)u_4(x)$.

Solution. **(a)** We associate a pair of the functions and apply the Derivative Product Rule:

$$\frac{d}{dx}[u_1(x)u_2(x)u_3(x)] = \frac{d}{dx}[(u_1(x)u_2(x))u_3(x)]$$

$$= \frac{d}{dx}[u_1(x)u_2(x)](u_3(x)) + (u_1(x)u_2(x))\frac{d}{dx}[u_3(x)]$$

by the Derivative Product Rule, Theorem 3.3.G

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Solution. (a) We associate a pair of the functions and apply the Derivative Product Rule:

$$\frac{d}{dx}[u_1(x)u_2(x)u_3(x)] = \frac{d}{dx}[(u_1(x)u_2(x))u_3(x)]$$

$$= \frac{d}{dx}[u_1(x)u_2(x)](u_3(x)) + (u_1(x)u_2(x))\frac{d}{dx}[u_3(x)]$$

by the Derivative Product Rule, Theorem 3.3.G

Exercise 3.3.77(a,b) (continued 1)

Solution (continued).

$$
\begin{aligned}
&= \frac{d}{dx}[u_1(x)u_2(x)](u_3(x)) + (u_1(x)u_2(x))\frac{d}{dx}[u_3(x)] \\
&= \left(\frac{d}{dx}[u_1(x)](u_2(x)) + (u_1(x))\frac{d}{dx}[u_2(x)]\right)(u_3(x)) \\
&\quad + (u_1(x)u_2(x))\frac{d}{dx}[u_3(x)] \\
&= \left([u_1'(x)](u_2(x)) + (u_1(x))[u_2'(x)]\right)(u_3(x)) + (u_1(x))(u_2(x))[u_3'(x)] \\
&= [u_1'(x)](u_2(x))(u_3(x)) + (u_1(x))[u_2'(x)](u_3(x)) + (u_1(x))(u_2(x))[u_3'(x)].
\end{aligned}
$$

Exercise 3.3.77(a,b) (continued 2)

Solution (continued). So, in a simplified notation that suppresses the variable x and uses the square brackets,

$$\frac{d}{dx}[u_1 u_2 u_3] = [u_1'](u_2)(u_3) + (u_1)[u_2'](u_3) + (u_1)(u_2)[u_3']. \quad \square$$

(b) We associate the first three functions, and apply the Derivative Product Rule, Theorem 3.3.G, and part (a):

$$\frac{d}{dx}[u_1(x)u_2(x)u_3(x)u_4(x)] = \frac{d}{dx}[(u_1(x)u_2(x)u_3(x))u_4(x)]$$

$$= \frac{d}{dx}[u_1(x)u_2(x)u_3(x)](u_4(x)) + (u_1(x)u_2(x)u_3(x))\frac{d}{dx}[u_4(x)]$$

by the Derivative Product Rule, Theorem 3.3.G

$$= \big[[u_1'(x)](u_2(x))(u_3(x)) + (u_1(x))[u_2'(x)](u_3(x))$$
$$+(u_1(x))(u_2(x))[u_3'(x)]\big]\,(u_4(x)) + (u_1(x)u_2(x)u_3(x))[u_4'(x)]$$

by part (a)

Exercise 3.3.77(a,b) (continued 3)

Solution (continued). ...

$$= [u_1'(x)](u_2(x))(u_3(x))(u_4(x)) + (u_1(x))[u_2'(x)](u_3(x))(u_4(x)) + (u_1(x))(u_2(x))[u_3'(x)](u_4(x)) + (u_1(x))(u_2(x))(u_3(x))[u_4'(x)].$$

So, in a simplified notation that suppresses the variable x and uses the square brackets, $\frac{d}{dx}[u_1 u_2 u_3 u_4] =$

$$[u_1'](u_2)(u_3)(u_4) + (u_1)[u_2'](u_3)(u_4) + (u_1)(u_2)[u_3'](u_4) + (u_1)(u_2)(u_3)[u_4'].$$

□

Exercise 3.3.42

Exercise 3.3.42. Find the derivatives of all orders of $y = \dfrac{x^5}{120}$.

Solution. We have $\dfrac{d}{dx}\left[\dfrac{x^5}{120}\right] = \dfrac{1}{120}[5x^4] = \boxed{\dfrac{x^4}{24} = y'}$.

Next, $\dfrac{d}{dx}\left[\dfrac{x^4}{24}\right] = \dfrac{1}{24}[4x^3] = \boxed{\dfrac{x^3}{6} = y''}$.

Then, $\dfrac{d}{dx}\left[\dfrac{x^3}{6}\right] = \dfrac{1}{6}[3x^2] = \boxed{\dfrac{x^2}{2} = y^{(3)}}$.

So, $\dfrac{d}{dx}\left[\dfrac{x^2}{2}\right] = \dfrac{1}{2}[2x] = \boxed{x = y^{(4)}}$.

Therefore, $\dfrac{d}{dx}[x] = \boxed{1 = y^{(5)}}$.

Hence, $y^{(6)} = \dfrac{d}{dx}[1] = 0$ and so $\boxed{y^{(n)} = 0 \text{ for all } n \geq 6}$. □

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Hence, $y^{(6)} = \dfrac{d}{dx}[1] = 0$ and so $\boxed{y^{(n)} = 0 \text{ for all } n \geq 6}$. $\square$

Exercise 3.3.66

Exercise 3.3.66. Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the graph of $F(x) = \dfrac{f(x)+3}{x-g(x)}$ at $x = 2$.

Solution. First, we find $F'(x)$ using the Derivative Quotient Rule, Theorem 3.3.H:

$$F'(x) = \frac{d}{dx}\left[\frac{f(x)+3}{x-g(x)}\right] = \frac{[f'(x)+0](x-g(x)) - (f(x)+3)[1-g'(x)]}{(x-g(x))^2}$$

$$= \frac{f'(x)(x-g(x)) - (f(x)+3)(1-g'(x))}{(x-g(x))^2}.$$

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$$= \frac{f'(x)(x-g(x)) - (f(x)+3)(1-g'(x))}{(x-g(x))^2}.$$

So when $x = 2$, $F'(2) = \dfrac{f'(2)((2)-g(2)) - (f(2)+3)(1-g'(2))}{((2)-g(2))^2} =$ $\dfrac{(-1)((2)-(-4)) - ((3)+3)(1-(1))}{((2)-(-4))^2} = \dfrac{-6}{36} = \dfrac{-1}{6}$.

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Exercise 3.3.66 (continued)

Exercise 3.3.66. Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the graph of $F(x) = \dfrac{f(x)+3}{x-g(x)}$ at $x = 2$.

Solution (continued). Now $F'(2) = -1/6$ is the slope of a line *tangent* to the graph $y = F(x)$ at $x = 2$, so the slope of a line *perpendicular* to $y = F(x)$ is the negative reciprocal of $F'(2) = -1/6$ and so the slope of the desired line is $m = 6$. Also, $F(2) = \dfrac{f(2)+3}{(2)-g(2)} = \dfrac{(3)+3}{(2)-(-4)} = \dfrac{6}{6} = 1$ so that a point on the desired line is $(x_1, y_1) = (2, F(2)) = (2, 1)$. From the point-slope equation of a line, the desired line is $y - y_1 = m(x - x_1)$ or $y - (1) = (6)(x - (2))$ or $\boxed{y = 6x - 11}$. □

Exercise 3.3.66 (continued)

Exercise 3.3.66. Assume that functions f and g are differentiable with $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, and $g'(2) = 1$. Find an equation of the line perpendicular to the graph of $F(x) = \dfrac{f(x)+3}{x - g(x)}$ at $x = 2$.

Solution (continued). Now $F'(2) = -1/6$ is the slope of a line *tangent* to the graph $y = F(x)$ at $x = 2$, so the slope of a line *perpendicular* to $y = F(x)$ is the negative reciprocal of $F'(2) = -1/6$ and so the slope of the desired line is $m = 6$. Also, $F(2) = \dfrac{f(2)+3}{(2) - g(2)} = \dfrac{(3)+3}{(2)-(-4)} = \dfrac{6}{6} = 1$ so that a point on the desired line is $(x_1, y_1) = (2, F(2)) = (2, 1)$. From the point-slope equation of a line, the desired line is $y - y_1 = m(x - x_1)$ or $y - (1) = (6)(x - (2))$ or $\boxed{y = 6x - 11}$. □

Exercise 3.3.80

Exercise 3.3.80. The Best Quantity to Order.
One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, TVs, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2.

Exercise 3.3.80 (continued)

Solution. We have $A(q) = \frac{km}{q} + cm + \frac{hq}{2}$, where q is the variable, and k, c, m, and h are constants. So, by the Derivative Quotient Rule, Theorem 3.3.H, $\frac{dA}{dq} = \frac{d}{dq}\left[\frac{km}{q} + cm + \frac{hq}{2}\right] = \frac{d}{dq}\left[\frac{km}{q}\right] + \frac{d}{dq}[cm] + \frac{d}{dq}\left[\frac{hq}{2}\right] = \frac{[0](q) - (km)[1]}{(q)^2} + [0] + \left[\frac{h}{2}\right] = \frac{-km}{q^2} + \frac{h}{2} = \boxed{-kmq^{-2} + \frac{h}{2}}$. Then by the Derivative Power Rule (General Version), Theorem 3.3.C, $\frac{d^2A}{dq^2} = \frac{d}{dq}\left[-kmq^{-2} + \frac{h}{2}\right] = -km[-2q^{-3}] + 0 = \boxed{2kmq^{-3}}$. □

Exercise 3.3.80 (continued)

Solution. We have $A(q) = \frac{km}{q} + cm + \frac{hq}{2}$, where q is the variable, and k, c, m, and h are constants. So, by the Derivative Quotient Rule, Theorem 3.3.H, $\frac{dA}{dq} = \frac{d}{dq}\left[\frac{km}{q} + cm + \frac{hq}{2}\right] = \frac{d}{dq}\left[\frac{km}{q}\right] + \frac{d}{dq}[cm] + \frac{d}{dq}\left[\frac{hq}{2}\right] = \frac{[0](q) - (km)[1]}{(q)^2} + [0] + \left[\frac{h}{2}\right] = \frac{-km}{q^2} + \frac{h}{2} = \boxed{-kmq^{-2} + \frac{h}{2}}$. Then by the Derivative Power Rule (General Version), Theorem 3.3.C, $\frac{d^2A}{dq^2} = \frac{d}{dq}\left[-kmq^{-2} + \frac{h}{2}\right] = -km[-2q^{-3}] + 0 = \boxed{2kmq^{-3}}$. $\square$