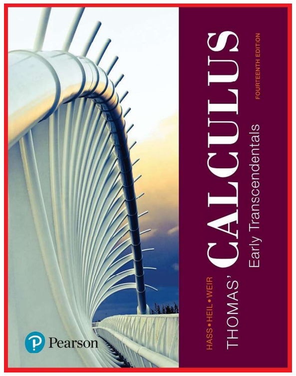
Calculus 1

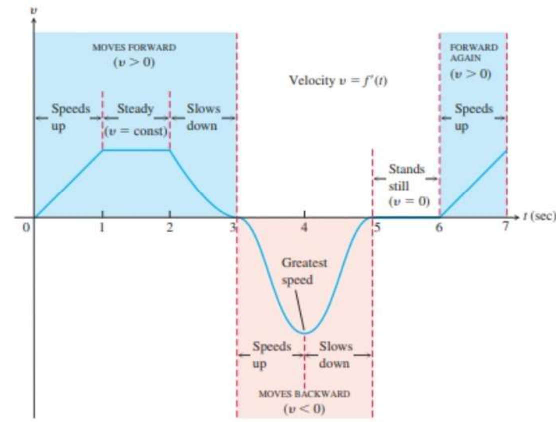
Chapter 3. Derivatives

3.4. The Derivative as a Rate of Change—Examples and Proofs

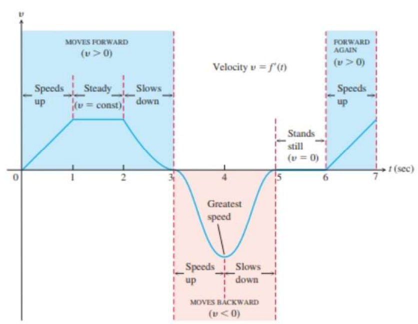


Example 3.4.2

Example 3.4.2. Figure 3.17 below shows the graph of the velocity $v = f'(t)$ of a particle moving along a horizontal line. Use this graph to discuss the movement of the particle at various time.

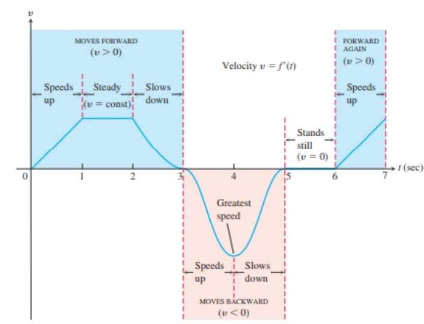


Example 3.4.2 (continued 1)



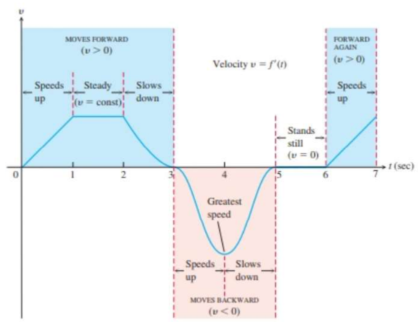
Solution. When velocity v is positive, $v > 0$, then the particle is moving forward and this happens for $t \in (0,3) \cup (6,7)$. When $v < 0$ the particle is moving backward and this happens for $t \in (3,5)$. On an interval for which $v = 0$, the particle is stationary and this happens for $t \in [5,6]$. Notice that the velocity is changing from $v > 0$ to $v < 0$ at $t = 3$, so the particle is moving forward up to $t = 3$, then is stops at $t = 3$, and starts moving backward for $3 < t < 5$.

Example 3.4.2 (continued 2)



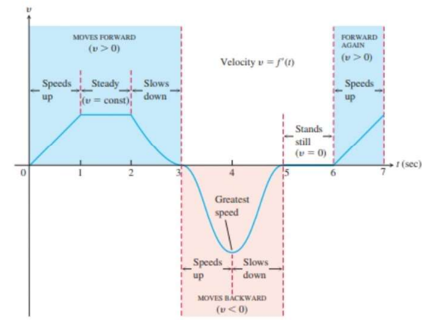
Solution. When velocity v is increasing and positive then the object is speeding up and this happens for $t \in (0,1) \cup (6,7)$. When v is constant then the object has constant speed and this happens for $t \in (1,2) \cup (5,6)$. Notice for $t \in (3,4)$, velocity decreases but gets larger in magnitude (i.e., gets larger in absolute value); so the particle is moving faster *backward* and so it is speeding up (even though v is decreasing).

Example 3.4.2 (continued 3)



Solution. When velocity v is decreasing and positive then the object is slowing down and this happens for $t \in (2, 3)$. Notice for $t \in (4, 5)$, velocity increases but gets smaller in magnitude (i.e., gets smaller in absolute value); so the particle is moving slower *backward* and so it is slowing down (even though v is increasing).

Example 3.4.2 (continued 4)



Solution. The rate of change of velocity v is called *acceleration* (as we will define shortly). So the particle is accelerating when v is increasing and this happens for $t \in (0, 1) \cup (4, 5) \cup (6, 7)$. The particle is decelerating when v is decreasing and this happens for $t \in (2, 3) \cup (3, 4)$. When v is constant, then acceleration is 0 and this happens for $t \in (1, 2) \cup (5, 6)$ (arguably, we could include the endpoints in all of these intervals). □

Exercise 3.4.12

Exercise 3.4.12. Speeding Bullet.
A 45-caliber bullet shot straight up from the surface of the moon would reach a height of $832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long will the bullet be aloft in each case? How high will the bullet go?

Solution. To find out how long the bullet is aloft, we set the height equal to 0. This will give two times: The time when the gun was fired and the time when the bullet has gone up and come back down to ground level again. On the surface of the moon we have $832t - 2.6t^2 = 0$ or $t(832 - 2.6t) = 0$ or $t = 0$ sec and $t = 832/2.6 = 320$ sec, so that the bullet is $\boxed{\text{aloft 320 sec on the moon}}$. On the surface of the Earth we have $832t - 16t^2 = 0$ or $t(832 - 16t) = 0$ or $t = 0$ sec and $t = 832/16 = 52$ sec, so that the bullet is $\boxed{\text{aloft 52 sec on the Earth}}$. (Notice that the ratio of these times is roughly 6 to 1; this is because the "the moon has 1/6 the gravity of the Earth.")

Exercise 3.4.12 (continued)

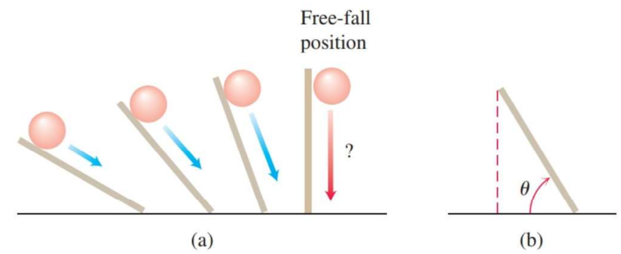
Solution. To find how high the bullet goes, we'll think like this: The bullet goes up (with positive velocity), stops (with 0 velocity), and falls down (with negative velocity). So we can find the time when the bullet is at its highest point by setting velocity equal to 0, and can then find the height at this time. On the moon, the velocity function is $v = \dfrac{d}{dt}[832t - 2.6t^2] = 832 - 5.2t$ ft/sec, so that the velocity is 0 ft/sec when $832 - 5.2t = 0$ or $t = 832/5.2 = 160$ sec; therefore the $\boxed{\text{maximum height on the moon is } 832(160) - 2.6(160)^2 = 66{,}560 \text{ ft}}$ (about 12.61 miles). On the Earth, the velocity function is $v = \dfrac{d}{dt}[832t - 16t^2] = 832 - 32t$ ft/sec, so that the velocity is 0 ft/sec when $832 - 32t = 0$ or $t = 832/32 = 26$ sec; therefore the $\boxed{\text{maximum height on the Earth is } 832(26) - 16(26)^2 = 10{,}816 \text{ ft}}$ (about 2.05 miles). (Notice that the maximum height is reached half way through the time that the bullet is aloft.) □

Exercise 3.4.14

Exercise 3.4.14. Galileo's Free-Fall Formula.
Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t sec into motion was a constant multiple of t. That is, the velocity was given by a formula of the form $y = kt$. The value of the constant k depended on the inclination of the plank. In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ, the ball's velocity t sec into the roll was $y = 9.8(\sin\theta)t$ m/sec.

Exercise 3.4.14 (continued 1)

(a) What is the equation for the ball's velocity during free fall? **(b)** Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?



Solution. (a) Since the velocity is $y = 9.8(\sin\theta)t$ m/sec, where θ is the angle the ramp makes with the horizontal, then for free-fall the ramp would be vertical and we would have $\theta = \pi/2$ so that $\sin\theta = \sin\pi/2 = 1$. So for free-fall, the velocity is $\boxed{y = 9.8t \text{ m/sec}}$. □

Exercise 3.4.14 (continued 2)

Solution. (b) Acceleration is

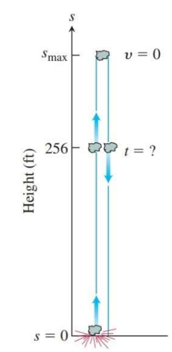
$$a(t) = \frac{dv}{dt} = \frac{d}{dt}[v] = \frac{d}{dt}[9.8t] = \boxed{9.8 \text{ m/sec}^2}.$$

Notice that the difference quotient associated with acceleration would be of the form $\dfrac{v(t+h) - v(t)}{h}$ where v is in units of m/sec and h is in units of sec so that acceleration has units of m/sec^2 (as expected). □

Note. So for an object in free-fall (at the surface of the Earth) we have that the acceleration is 9.8 m/sec^2. If we treat acceleration as a *vector quantity* with "up" as positive and "down" as negative, then we would have that the acceleration due to gravity at the surface of the Earth is -9.8 m/sec^2. If we measured distance in feet instead of meters, then we find that the acceleration due to gravity at the surface of the Earth is 32 ft/sec^2 (down); also, we would have that velocity in free-fall is $v = 32t$ ft/sec and the distance that the object falls is $16t^2$ ft, as claimed in Section 2.1.

Example 3.3.4

Example 3.3.4. A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.19a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.
(a) How high does the rock go?
(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
(c) What is the acceleration of the rock at any time t during its flight (after the blast)?
(d) When does the rock hit the ground again?



Solution. As in the previous example, we know that the rock goes up with positive velocity, reaches its maximum height when the velocity is 0, and then falls with negative velocity.

Example 3.3.4 (continued 1)

Example 3.3.4.
(a) How high does the rock go?
(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

Solution (continued). The velocity is $v = \frac{d}{dt}[160t - 16t^2] = 160 - 32t$ ft/sec, and $160 - 32t = 0$ ft/sec implies $t = 5$ sec. So the $\boxed{\text{maximum height is } s(5) = 160(5) - 16(5)^2 = 400 \text{ ft}}$.

(b) The rock is at a height of 256 ft when $160t - 16t^2 = 256$ or $16t^2 - 160t + 256 = 0$ or (dividing by 16) $t^2 - 10t + 16 = 0$ or $(t-2)(t-8) = 0$ or when $t = 2$ sec ("on the way up") and $t = 8$ sec ("on the way down"). At these times, the velocity is $\boxed{v(2) = 160 - 32(2) = 96 \text{ ft/sec on the way up}}$, and the velocity is $\boxed{v(8) = 160 - 32(8) = -96 \text{ ft/sec on the way down}}$. In both cases the $\boxed{\text{speed is 96 ft/sec}}$.

Example 3.3.4 (continued 2)

Example 3.3.4.
(c) What is the acceleration of the rock at any time t during its flight (after the blast)?
(d) When does the rock hit the ground again?

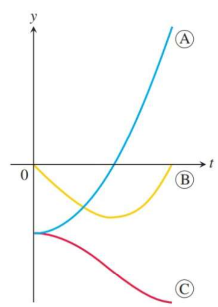
Solution (continued). (c) The acceleration is $\frac{d^2}{dt^2}[160t - 16t^2] = \frac{d}{dt}[160 - 32t] = \boxed{-32 \text{ ft/sec}^2}$.

(d) The rock hits the ground when the height is 0 ft, so we consider $160t - 16t^2 = 0$ or $16t(10 - t) = 0$, which implies that $t = 0$ sec and $t = 10$ sec. At 0 sec the blast goes off and $\boxed{\text{at 10 sec the rock hits the ground}}$. □

Note. Notice the symmetry in that the the speed is the same at height 256 ft whether going up or down, and that the maximum height occurs half way through the flight of the rock.

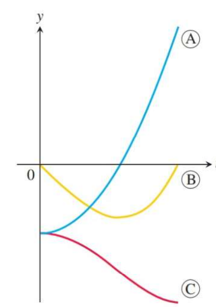
Exercise 3.4.22

Exercise 3.4.22. The graphs in the accompanying figure show the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.



Solution. We have that the rate of change of position s is velocity v, and the rate of change of velocity v is acceleration a. Notice that when A is negative then B is decreasing, and when A is positive then B is increasing. In addition, when A is 0 then B has a horizontal tangent line. Hence the rate of change of curve B is curve A.

Exercise 3.4.22 (continued)



Solution. Next, C is always decreasing and B is always negative. In addition, when B is 0 then C has a horizontal tangent. Hence, the rate of change of curve C is curve B. That is, the derivative of curve C is curve B, and the derivative of curve B is curve A. So we must have that $\boxed{\text{C is the graph of position } s}$, $\boxed{\text{B is the graph of velocity } v}$, and $\boxed{\text{A is the graph of acceleration } a}$. □

Exercise 3.4.24

Exercise 3.4.24. Marginal Revenue.
Suppose that the revenue from selling x washing machines is $r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$ dollars. **(a)** Find the marginal revenue when 100 machines are produced. **(b)** Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines per week to 101 machines per week. **(c)** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Solution. (a) The marginal revenue is

$$r'(x) = \frac{d}{dx}\left[20{,}000\left(1 - \frac{1}{x}\right)\right] = 20{,}000\frac{d}{dx}\left[1 - \frac{1}{x}\right] = 20{,}000\frac{d}{dx}\left[1 - x^{-1}\right]$$

$$= 20{,}000[-(-1)x^{-2}] = \frac{20{,}000}{x^2}.$$

Since x is revenue (measured in dollars) and x is the number of washing machines, then the units of r' is dollars/machine.

Exercise 3.4.24 (continued 1)

Exercise 3.4.24. Marginal Revenue.
(a) Find the marginal revenue when 100 machines are produced. **(b)** Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines per week to 101 machines per week.

Solution (continued). The marginal revenue at $x = 100$ machines is $r'(100) = \frac{20{,}000}{(100)^2} = \frac{20{,}000}{10{,}000} = \boxed{2 \text{ dollars/machine}}$.

(b) So if x increases from 100 machines per week to 101 machines per week (a change of $\Delta x = 1$ machine per week) then the change in revenue is approximately
$r'(100)\Delta x = (2 \text{ dollars/machine})(1 \text{ machine/week}) = \boxed{2 \text{ dollars/week}}$.

Exercise 3.4.24 (continued 2)

Exercise 3.4.24. Marginal Revenue.
(c) Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Solution (continued). (c) We have

$$\lim_{x\to\infty} r'(x) = \lim_{x\to\infty}\frac{20{,}000}{x^2} = 20{,}000\left(\lim_{x\to\infty} 1/x\right)^2$$

by the Constant Multiple Rule and Power Rule, Theorem 2.12(4,6)

$$= 20{,}000(0)^2 = \boxed{0}.$$

So as the number of washing machines x gets larger and larger, the revenue increases (since $r'(x) = 20{,}000/x^2 > 0$ for all $x > 0$) but the $\boxed{\text{change in revenue becomes less and less as } x \text{ gets larger}}$. In fact, we can verify that $r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$ has a horizontal asymptote of $y = \$20{,}000$.

Exercise 3.4.30

Exercise 3.4.30. Inflating a Balloon.
The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius. **(a)** At what rate (ft^3/ft) does the volume change with respect to the radius when $r = 2$ ft? **(b)** By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

Solution. (a) The rate of change of V with respect to r is $\frac{d}{dr}[V] = \frac{d}{dr}[(4/3)\pi r^3] = (4/3)\pi[3r^2] = 4\pi r^2$. Since V is measured in ft^3 and r is measured in ft, then $\frac{dV}{dr}$ is measured in ft^3/ft. When $r = 2$ ft, then $\left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = \boxed{16\pi \text{ ft}^3/\text{ft}}$.

Exercise 3.4.30 (continued)

Exercise 3.4.30. Inflating a Balloon.
The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius. **(a)** At what rate (ft^3/ft) does the volume change with respect to the radius when $r = 2$ ft? **(b)** By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

Solution. (b) For r changing from $r = 2$ ft to $r = 2.2$ ft, we have the change in r of $\Delta r = 2.2 - 2 = 0.2$ ft. So the change in V corresponding to this change in r is approximately the instantaneous change in V at $r = 2$ times the change Δr of r:

$$\left(\left.\frac{dV}{dr}\right|_{r=2}\right)\Delta r = (16\pi\ \text{ft}^3/\text{ft})(0.2\ \text{ft}) = \boxed{3.2\pi\ \text{ft}^3}.$$

□