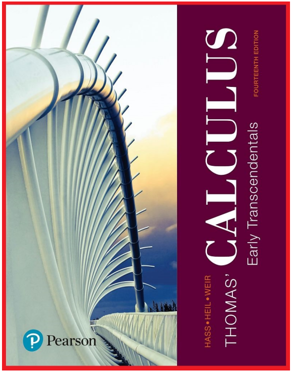
Calculus 1

Chapter 3. Derivatives

3.5. Derivatives of Trigonometric Functions—Examples and Proofs



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Theorem 3.5.A

Theorem 3.5.A. Derivative of the Sine Function

$$\frac{d}{dx}[\sin x] = \cos x$$

Proof. Let $y = \sin x$. By definition we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \text{ by the summation formula} \\
&= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \frac{\sin h}{h} \right) \\
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Theorem 3.5.A (continued)

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$$\frac{d}{dx}[\sin x] = \cos x$$

Proof (continued). ...

$$\begin{aligned}\frac{dy}{dx} &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\ &= (\sin x)(0) + (\cos x)(1) \\ &= \cos x.\end{aligned}$$

We have $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h\to 0} \frac{\sin h}{h} = 1$ by the results in Section 2.4. □

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Exercise 3.5.2

Exercise 3.5.2. Differentiate $y = \dfrac{3}{x} + 5\sin x$.

Solution. First, let $y = \dfrac{3}{x} + 5\sin x = 3x^{-1} + 5\sin x$. Then we have $y' = 3[-x^{-2}] + 5[\cos x] = \boxed{-3x^{-2} + 5\cos x}$. □

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Theorem 3.5.B

Theorem 3.5.B. Derivative of the Cosine Function

$$\frac{d}{dx}[\cos x] = -\sin x$$

Proof. Let $y = \cos x$. By definition we have

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \text{ by the summation formula} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \frac{\sin h}{h} \\
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Theorem 3.5.B (continued)

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$$\frac{d}{dx}[\cos x] = -\sin x$$

Proof (continued). ...

$$\begin{aligned} \frac{dy}{dx} &= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} \\ &= (\cos x)(0) - (\sin x)(1) \\ &= -\sin x. \end{aligned}$$

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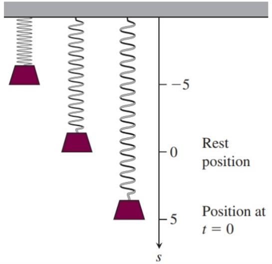
Example 3.5.3

Example 3.5.3. Simple Harmonic Motion.
A weight hanging from a spring is stretched down 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is $s = 5\cos t$. What are its velocity and acceleration at time t.



Solution. Since the position $s = 5\cos t$, then the velocity is $v = ds/dt = -5\sin t$ and the acceleration is $a = dv/dt = -5\cos t$.

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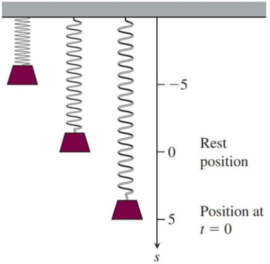

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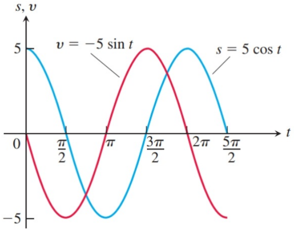

Solution (continued). Here, position is graphed in blue and velocity is graphed in red. Notice that weight oscillates up and down with a period of 2π. Initially, the weight has decreasing s value for $t \in (0, \pi)$ and the velocity is negative for these t values; the velocity is a minimum at $t = \pi/2$ and the *speed* is a maximum then (when the weight is at the center position about which it oscillates).

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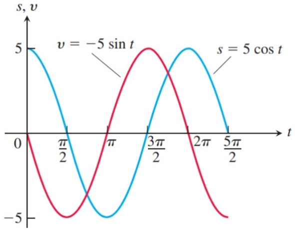

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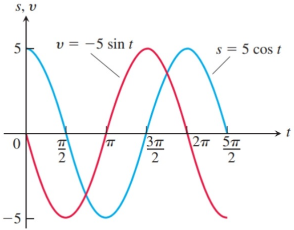

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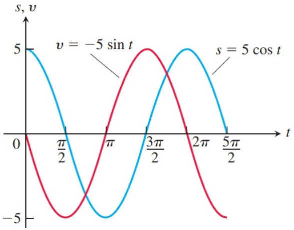
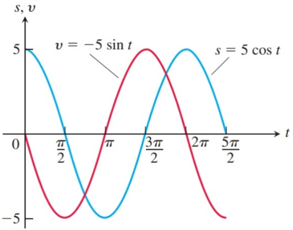
Example 3.5.3 (continued 2)



Solution (continued). At the t values $0, \pi, 2\pi$ the velocity is 0; at these points the weight has reached its maximum displacement from equilibrium, it has stopped, and is about to reverse direction. The acceleration $a = -5\cos t$ of the weight is always proportional to the negative of its displacement $-s = -5\cos t$. This is called *Hooke's Law* and the motion is called *simple harmonic motion*. □

Example 3.5.3 (continued 2)



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Exercise 3.5.28

Exercise 3.5.28. Find dp/dq when $p = (1 + \csc q)\cos q$. Use the square bracket notation.

Solution. By the Derivative Product Rule (Theorem 3.3.G) and the fact that $\csc q = \dfrac{1}{\sin q}$ we have:

$$\begin{aligned}
\frac{dp}{dq} &= \frac{d}{dq}[(1+\csc q)\cos q] = \frac{d}{dq}\left[\left(1+\frac{1}{\sin q}\right)\cos q\right] \\
&= \left[0 + \frac{[0](\sin q) - (1)[\cos q]}{(\sin q)^2}\right](\cos q) + \left(1 + \frac{1}{\sin q}\right)[-\sin q] \\
&\quad \text{by the Derivative Product Rule (Theorem 3.3.G)} \\
&\quad \text{and the Derivative Quotient Rule (Theorem 3.3.H)} \\
&= \left[0 - \frac{\cos q}{(\sin q)^2}\right](\cos q) + \left(-\sin q - \frac{\sin q}{\sin q}\right)
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Exercise 3.5.28 (continued)

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Solution (continued).

$$\begin{aligned}
\frac{dp}{dq} &= \left[0 - \frac{\cos q}{(\sin q)^2}\right](\cos q) + \left(-\sin q - \frac{\sin q}{\sin q}\right) \\
&= \left[0 - \frac{1}{\sin q}\frac{\cos q}{\sin q}\right](\cos q) + (-\sin q - 1) \\
&= [-\csc q \cot q](\cos q) + (-\sin q - 1) \qquad (*) \\
&= \boxed{-\csc q \cot q \cos q - \sin q - 1}.
\end{aligned}$$

Notice from $(*)$ that the quantity in square brackets corresponds to the derivative of $\csc q$, so that we have shown $\frac{d}{dx}[\csc x] = -\csc x \cot x$. □

Example 3.5.5

Example 3.5.5. Find $\dfrac{d}{dx}[\tan x]$.

Solution. We use the Derivative Quotient Rule (Theorem 3.3.H):

$$\begin{aligned}\frac{d}{dx}[\tan x] &= \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{[\cos x](\cos x) - (\sin x)[-\sin x]}{(\cos x)^2} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x},\end{aligned}$$

since $\sec x = 1/\cos x$. □

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Exercise 3.5.62

Exercise 3.5.62. Derive the formula for the derivative with respect to x of **(a)** $\sec x$ and **(c)** $\cot x$.

Solution. **(a)** We use the Derivative Quotient Rule (Theorem 3.3.H):

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{[0](\cos x) - (1)[-\sin x]}{(\cos x)^2}$$
$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \boxed{\sec x \tan x},$$

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Exercise 3.5.62 (continued)

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Solution (continued). (a) We use the Derivative Quotient Rule (Theorem 3.3.H):

$$\begin{aligned} \frac{d}{dx}[\cot x] &= \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{[-\sin x](\sin x) - (\cos x)[\cos x]}{(\sin x)^2} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}, \end{aligned}$$

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Exercise 3.5.40

Exercise 3.5.40. Does the graph of $y = 2x + \sin x$ have any horizontal tangent lines in the interval $0 \leq x \leq 2\pi$.

Solution. First, $y' = 2 + \cos x$. A tangent line to the graph of $y = f(x)$ has slope $y' = 2 + \cos x$ as a function of x and a horizontal line has slope has slope 0, so we look for x values where $y' = 2 + \cos x = 0$.

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Exercise 3.5.50

Exercise 3.5.50. Evaluate $\lim_{\theta \to \pi/4} \frac{\tan\theta - 1}{\theta - \pi/4}$.

Solution. This one requires a trick. Recall the Alternative Formula for the Derivative: $f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$ (see Exercise 3.2.24). With $z = \theta$, $x = \pi/4$ and $f(x) = \tan x$ (so that $\tan \pi/4 = 1$), we have

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{\theta \to \pi/4} \frac{\tan\theta - 1}{\theta - \pi/4}.$$

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$$f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\pi/4}.$$

Since $f(x)=\tan x$ then $f'(x)=\sec^2 x$. Hence

$$\lim_{\theta\to\pi/4}\frac{\tan\theta-1}{\theta-\pi/4}=\sec^2\pi/4=\frac{1}{\cos^2\pi/4}=\frac{1}{(\sqrt{2}/2)^2}=\left(\frac{2}{\sqrt{2}}\right)^2=\boxed{2}.$$

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Since $f(x) = \tan x$ then $f'(x) = \sec^2 x$. Hence

$$\lim_{\theta\to\pi/4} \frac{\tan\theta - 1}{\theta - \pi/4} = \sec^2\pi/4 = \frac{1}{\cos^2\pi/4} = \frac{1}{(\sqrt{2}/2)^2} = \left(\frac{2}{\sqrt{2}}\right)^2 = \boxed{2}.$$

□