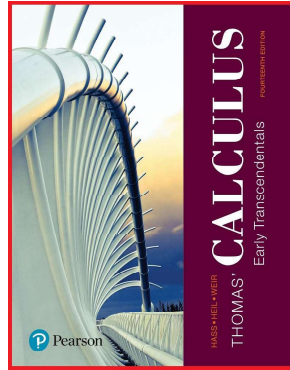


Calculus 1

Chapter 3. Derivatives

3.6. The Chain Rule—Examples and Proofs



Theorem 3.2

Theorem 3.2. The Chain Rule. (Proof of a Special Case.)

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , AND there is some $\varepsilon > 0$ such that $\Delta u = g(x + \Delta x) - g(x) \neq 0$ for all x in the domain of g and for all $\Delta x < \varepsilon$ THEN the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x))[g'(x)].$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Proof. Let $\varepsilon > 0$. Let $0 < \Delta x < \varepsilon$. Let Δu be the change in u when x changes by $\Delta x < \varepsilon$, so that $\Delta u = g(x + \Delta x) - g(x)$ and $\Delta u \neq 0$ (by the choice of Δx and the "special case" hypotheses).

Theorem 3.2 (continued 1)

Proof (continued). Since y is a function of u , then the change in y that results when x changes by an amount Δx is $\Delta y = f(u + \Delta u) - f(u)$. Since $\Delta u \neq 0$ (*this* is where we use the special case hypotheses) then we can write the fraction $\Delta y/\Delta x$ as $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x}$. Now $\Delta y/\Delta u$ is a difference quotient for function y with increment Δu . So $\frac{dy}{du} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}$. Notice that u is hypothesized to be differentiable at x , so by Theorem 3.1 (Differentiability Implies Continuity), u is continuous at x and so

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \Delta u &= \lim_{\Delta x \rightarrow 0} g(x + \Delta x) - g(x) \\ &= g\left(\lim_{\Delta x \rightarrow 0} (x + \Delta x)\right) - g(x) = g(x + 0) - g(x) = 0 \end{aligned}$$

(by Theorem 2.10. Limits of Continuous Functions). That is, $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$.

Theorem 3.2 (continued 2)

Proof (continued). Therefore

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta u}\right) \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x}\right) \\ &= \lim_{\Delta u \rightarrow 0} \left(\frac{\Delta y}{\Delta u}\right) \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x}\right) \text{ since } \Delta u \rightarrow 0 \text{ as } \Delta x \rightarrow 0, \\ &\quad \text{as shown above} \\ &= \frac{dy}{du} \frac{du}{dx}, \end{aligned}$$

as claimed. □

Exercise 3.6.8

Exercise 3.6.8. Given $y = -\sec u$ and $u = g(x) = \frac{1}{x} + 7x$, find $\frac{dy}{dx} = f'(g(x))g'(x)$. Use the square bracket and little arrow notation.

Solution. By the Chain Rule (Theorem 3.2), $\frac{dy}{dx} = \frac{dy}{du} \left[\frac{du}{dx} \right]$. Now

$$\frac{dy}{du} = \frac{d}{du}[-\sec u] = -[\sec u \tan u] = -\sec u \tan u \text{ and}$$

$$\frac{du}{dx} = \frac{d}{dx} \left[\frac{1}{x} + 7x \right] = \frac{d}{dx} [x^{-1} + 7x] = [-x^{-2} + 7]. \text{ So}$$

$$\frac{dy}{dx} = \frac{dy}{du} \left[\frac{du}{dx} \right] = -\sec u \tan u \left[-x^{-2} + 7 \right]$$

$$= \boxed{-\sec \left(\frac{1}{x} + 7x \right) \tan \left(\frac{1}{x} + 7x \right) \left[-\frac{1}{x^2} + 7 \right]}. \quad \square$$

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Exercise 3.6.48

Exercise 3.6.48. Find the derivative of $q = \cot \left(\frac{\sin t}{t} \right)$. Use the square bracket and little arrow notation.

Solution. Since the derivative of $\cot x$ is $-\csc^2 x$, then by the Chain Rule (Theorem 3.2) and the Derivative Quotient Rule (Theorem 3.3.H) we have

$$\frac{dq}{dt} = \frac{d}{dt} \left[\cot \left(\frac{\sin t}{t} \right) \right] = \boxed{-\csc^2 \left(\frac{\sin t}{t} \right) \left[\frac{[\cos t](t) - (\sin t)[1]}{(t)^2} \right]}.$$

□

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Exercise 3.6.64

Exercise 3.6.64. Find dy/dt when $y = \frac{1}{6}(1 + \cos^2(7t))^3$. Use the square bracket and little arrow notation.

Solution. We have four “levels” of functions. The $7t$ function is inside the cosine function, the cosine function is inside the squaring function (plus 1), and this is inside the cubing function (times $1/6$). So we will have to use the Chain Rule (Theorem 3.2) three times. We have

$$\frac{dy}{dt} = \frac{d}{dt} \left[\frac{1}{6}(1 + \cos^2(7t))^3 \right] = \frac{1}{6} [3(1 + \cos^2(7t))^2 [0 + 2 \cos(7t) [-\sin(7t) [7]]]]$$

$$= \boxed{-7(1 + \cos^2(7t)) \cos(7t) \sin(7t)}.$$

□

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Exercise 3.6.88

Exercise 3.6.88. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

Solution. By the Chain Rule (Theorem 3.2),

$$\frac{dr}{dt} = \frac{d}{dt} [\sin(f(t))] = \cos(f(t)) \left[f'(t) \right]. \text{ So when } t = 0, \text{ we have}$$

$$\left. \frac{dr}{dt} \right|_{t=0} = \cos(f(0))(f'(0)) = \cos(\pi/3)(4) = \left(\frac{1}{2} \right) (4) = \boxed{2}.$$

□

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Exercise 3.6.96

Exercise 3.6.96. Find the equation of the line tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

Solution. The slope of the line is the derivative dy/dx evaluated at $x = 2$. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\sqrt{x^2 - x + 7} \right] = \frac{d}{dx} \left[(x^2 - x + 7)^{1/2} \right] \\ &= \frac{1}{2} (x^2 - x + 7)^{-1/2} [2x - 1] = \frac{2x - 1}{2\sqrt{x^2 - x + 7}}.\end{aligned}$$

So the slope of the desired line is

$$m = \left. \frac{dy}{dx} \right|_{x=2} = \frac{2(2) - 1}{2\sqrt{(2)^2 - (2) + 7}} = \frac{3}{2\sqrt{9}} = \frac{3}{2}.$$

Since the line contains the point $(x_1, y_1) = (2, \sqrt{(2)^2 - (2) + 7}) = (2, 3)$, then by the point slope formula for a line, the desired line is $y - y_1 = m(x - x_1)$ or

$$y - (3) = (1/2)(x - 2) \text{ or } \boxed{y = (1/2)x + 2}. \quad \square$$

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Exercise 3.6.58

Exercise 3.6.58. Find dy/dt when $y = \left(e^{\sin(t/2)} \right)^3$. Use the square bracket and little arrow notation.

Solution. We have four “levels” of functions. The $t/2$ function is inside the sine function, the sine function is inside the exponential function, and this is inside the cubing function. So we will have to use the Chain Rule (Theorem 3.2) three times. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\left(e^{\sin(t/2)} \right)^3 \right] = 3 \left(e^{\sin(t/2)} \right)^2 \left[e^{\sin(t/2)} \left[\cos(t/2) \left[1/2 \right] \right] \right] \\ &= \boxed{\frac{3}{2} \left(e^{\sin(t/2)} \right)^3 \cos(t/2)}.\end{aligned}$$

□

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Exercise 3.6.104

Exercise 3.6.104. Particle Acceleration.

A particle moves along the x -axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

Solution. The acceleration is the derivative of velocity with respect to time, so

$$\begin{aligned}a &= \frac{d}{dt} \left[\frac{dx}{dt} \right] = \frac{d}{dt} [f(x)] = \frac{d}{dx} [f(x)] \left[\frac{dx}{dt} \right] \\ &= f'(x) \frac{dx}{dt} = f'(x) f(x) = f(x) f'(x).\end{aligned}$$

as claimed. □

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