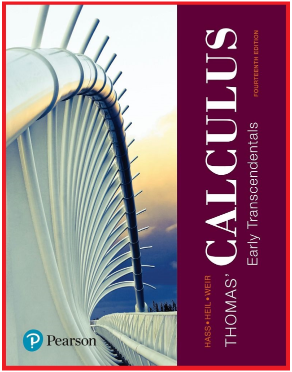
Calculus 1

Chapter 3. Derivatives

3.6. The Chain Rule—Examples and Proofs



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Theorem 3.2

Theorem 3.2. The Chain Rule. (Proof of a Special Case.)
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, AND there is some $\varepsilon > 0$ such that $\Delta u = g(x + \Delta x) - g(x) \neq 0$ for all x in the domain of g and for all $\Delta x < \varepsilon$ THEN the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))[g'(x)].$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Proof. Let $\varepsilon > 0$. Let $0 < \Delta x < \varepsilon$. Let Δu be the change in u when x changes by $\Delta x < \varepsilon$, so that $\Delta u = g(x + \Delta x) - g(x)$ and $\Delta u \neq 0$ (by the choice of Δx and the "special case" hypotheses).

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Theorem 3.2 (continued 1)

Proof (continued). Since y is a function of u, then the change in y that results when x changes by an amount Δx is $\Delta y = f(u + \Delta u) - f(u)$. Since $\Delta u \neq 0$ (*this* is where we use the special case hypotheses) then we can write the fraction $\Delta y/\Delta x$ as $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u}\dfrac{\Delta u}{\Delta x}$. Now $\Delta y/\Delta x$ is a difference quotient for function y with increment Δx. So $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

Notice that u is hypothesized to be differentiable at x, so by Theorem 3.1 (Differentiability Implies Continuity), u is continuous at x and so

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} g(x + \Delta x) - g(x)$$

$$= g\left(\lim_{\Delta x \to 0}(x + \Delta x)\right) - g(x) = g(x + 0) - g(x) = 0$$

(by Theorem 2.10. Limits of Continuous Functions). That is, $\Delta u \to 0$ as $\Delta x \to 0$.

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Proof (continued). Since y is a function of u, then the change in y that results when x changes by an amount Δx is $\Delta y = f(u + \Delta u) - f(u)$. Since $\Delta u \neq 0$ (*this* is where we use the special case hypotheses) then we can write the fraction $\Delta y/\Delta x$ as $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u}\dfrac{\Delta u}{\Delta x}$. Now $\Delta y/\Delta x$ is a difference quotient for function y with increment Δx. So $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. Notice that u is hypothesized to be differentiable at x, so by Theorem 3.1 (Differentiability Implies Continuity), u is continuous at x and so

$$\lim_{\Delta x \to 0} \Delta u = \lim_{\Delta x \to 0} g(x + \Delta x) - g(x)$$

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Theorem 3.2 (continued 2)

Proof (continued). Therefore

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u}\right) \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x}\right) \\
&= \lim_{\Delta u \to 0} \left(\frac{\Delta y}{\Delta u}\right) \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x}\right) \text{ since } \Delta u \to 0 \text{ as } \Delta x \to 0, \\
&\quad \text{as shown above} \\
&= \frac{dy}{du}\frac{du}{dx},
\end{aligned}$$

as claimed. □

Exercise 3.6.8

Exercise 3.6.8. Given $y = -\sec u$ and $u = g(x) = \dfrac{1}{x} + 7x$, find $\dfrac{dy}{dx} = f'(g(x))g'(x)$. Use the square bracket and little arrow notation.

Solution. By the Chain Rule (Theorem 3.2), $\dfrac{dy}{dx} = \dfrac{dy}{du}\left[\dfrac{du}{dx}\right]$. Now

$$\frac{dy}{du} = \frac{d}{du}[-\sec u] = -[\sec u \tan u] = -\sec u \tan u \text{ and}$$

$$\frac{du}{dx} = \frac{d}{dx}\left[\frac{1}{x} + 7x\right] = \frac{d}{dx}[x^{-1} + 7x] = [-x^{-2} + 7].$$

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$\frac{du}{dx} = \frac{d}{dx}\left[\frac{1}{x} + 7x\right] = \frac{d}{dx}[x^{-1} + 7x] = [-x^{-2} + 7]$. So

$$\frac{dy}{dx} = \frac{dy}{du}\left[\frac{du}{dx}\right] = -\sec u \tan u \left[-x^{-2} + 7\right]$$

$$= \boxed{-\sec\left(\frac{1}{x} + 7x\right)\tan\left(\frac{1}{x} + 7x\right)\left[-\frac{1}{x^2} + 7\right]}. \qquad \square$$

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Exercise 3.6.48

Exercise 3.6.48. Find the derivative of $q = \cot\left(\dfrac{\sin t}{t}\right)$. Use the square bracket and little arrow notation.

Solution. Since the derivative of $\cot x$ is $-\csc^2 x$, then by the Chain Rule (Theorem 3.2) and the Derivative Quotient Rule (Theorem 3.3.H) we have

$$\frac{dq}{dt} = \frac{d}{dt}\left[\cot\left(\frac{\sin t}{t}\right)\right] = \boxed{-\csc^2\left(\frac{\sin t}{t}\right)^{\curvearrowright}\left[\frac{[\cos t](t) - (\sin t)[1]}{(t)^2}\right]}.$$

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Exercise 3.6.64

Exercise 3.6.64. Find dy/dt when $y = \frac{1}{6}(1 + \cos^2(7t))^3$. Use the square bracket and little arrow notation.

Solution. We have four "levels" of functions. The $7t$ function is inside the cosine function, the cosine function is inside the squaring function (plus 1), and this is inside the cubing function (times $1/6$). So we will have to use the Chain Rule (Theorem 3.2) three times.

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$$\frac{dy}{dt} = \frac{d}{dt}\left[\frac{1}{6}(1+\cos^2(7t))^3\right] = \frac{1}{6}[3(1+\cos^2(7t))^2[0+2\cos(7t)[-\sin(7t)[7]]]]$$

$$= \boxed{-7(1+\cos^2(7t))\cos(7t)\sin(7t)}.$$

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Exercise 3.6.88

Exercise 3.6.88. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

Solution. By the Chain Rule (Theorem 3.2), $\frac{dr}{dt} = \frac{d}{dt}[\sin(f(t))] = \cos(f(t))[f'(t)]$. So when $t = 0$, we have

$$\left.\frac{dr}{dt}\right|_{t=0} = \cos(f(0))(f'(0)) = \cos(\pi/3)(4) = \left(\frac{1}{2}\right)(4) = \boxed{2}.$$

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Exercise 3.6.96. Find the equation of the line tangent to $y = \sqrt{x^2 - x + 7}$ at $x = 2$.

Solution. The slope of the line is the derivative dy/dx evaluated at $x = 2$. We have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sqrt{x^2 - x + 7}\right] = \frac{d}{dx}\left[(x^2 - x + 7)^{1/2}\right]$$

$$= \frac{1}{2}(x^2 - x + 7)^{-1/2}[2x - 1] = \frac{2x - 1}{2\sqrt{x^2 - x + 7}}.$$

So the slope of the desired line is

$$m = \left.\frac{dy}{dx}\right|_{x=2} = \frac{2(2) - 1}{2\sqrt{(2)^2 - (2) + 7}} = \frac{3}{2\sqrt{9}} = \frac{1}{2}.$$

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Exercise 3.6.58. Find dy/dt when $y = \left(e^{\sin(t/2)}\right)^3$. Use the square bracket and little arrow notation.

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$$\frac{dy}{dx} = \frac{d}{dx}\left[\left(e^{\sin(t/2)}\right)^3\right] = 3\left(e^{\sin(t/2)}\right)^2 [e^{\sin(t/2)}[\cos(t/2)[1/2]]]$$

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Exercise 3.6.104

Exercise 3.6.104. Particle Acceleration.
A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

Solution. The acceleration is the derivative of velocity with respect to time, so

$$a = \frac{d}{dt}\left[\frac{dx}{dt}\right] = \frac{d}{dt}[f(x)] = \frac{d}{dx}[f(x)]\left[\frac{dx}{dt}\right]$$
$$= f'(x)\frac{dx}{dt} = f'(x)f(x) = f(x)f'(x).$$

as claimed. $\square$

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