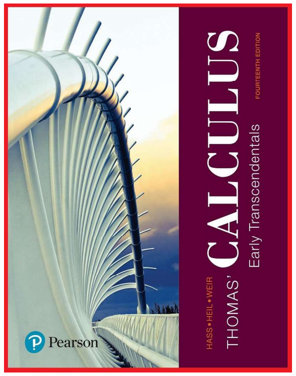
Calculus 1

Chapter 3. Derivatives

3.7. Implicit Differentiation—Examples and Proofs



Example 3.7.A

Example 3.7.A. Find the slope of the line tangent to $x^2 + y^2 = 1$ at $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. Do the same for the point $(x, y) = (\sqrt{2}/2, -\sqrt{2}/2)$.

Solution. We have just seen that implicit differentiation gives $dy/dx = -x/y$. So at $(\sqrt{2}/2, \sqrt{2}/2)$, the slope of a line tangent to $x^2 + y^2 = 1$ is

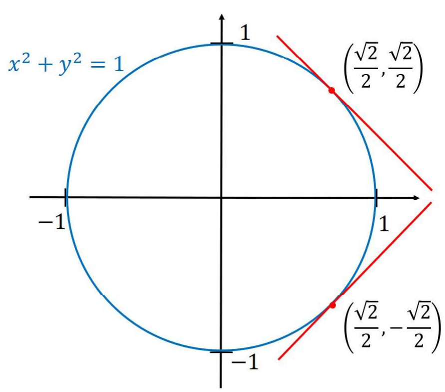
$$\left.\frac{dy}{dx}\right|_{(x,y)=(\sqrt{2}/2,\sqrt{2}/2)} = -\frac{(\sqrt{2}/2)}{(\sqrt{2}/2)} = \boxed{-1}.$$

At $(\sqrt{2}/2, -\sqrt{2}/2)$, the slope of a line tangent to $x^2 + y^2 = 1$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\sqrt{2}/2,-\sqrt{2}/2)} = -\frac{(\sqrt{2}/2)}{(-\sqrt{2}/2)} = \boxed{1}.$$

Example 3.7.A (continued)

Solution (continued).



Since $x^2 + y^2 = 1$ does not determine a single function (it fails the vertical line test), then to find the slope of a tangent to the graph of the equation, we need both an x value and a y for the point of tangency. □

Exercise 3.7.16

Exercise 3.7.16. Find dy/dx for y an implicit function of x given by the equation $e^{x^2y} = 2x + 2y$.

Solution. Differentiating implicitly, $\frac{d}{dx}[e^{x^2y}] = \frac{d}{dx}[2x + 2y]$ or $e^{x^2y}[[2x](y) + (x^2)[dy/dx]] = 2[1] + 2[dy/dx]$ (notice that since y is a function of x then we must use the Derivative Product Rule to differentiate x^2y). Solving for dy/dx we get $e^{x^2y}(2xy + x^2(dy/dx)) = 2 + 2(dy/dx)$ or $2xye^{x^2y} + x^2e^{x^2y}(dy/dx) = 2 + 2(dy/dx)$ or

$x^2e^{x^2y}(dy/dx) - 2(dy/dx) = -2xye^{x^2y} + 2$ or $\boxed{\frac{dy}{dx} = \frac{-2xye^{x^2y} + 2}{x^2e^{x^2y} - 2}}$. □

Exercise 3.7.20

Exercise 3.7.20. Find $dr/d\theta$ for r an implicit function of θ given by the equation $\cos r + \cot\theta = e^{r\theta}$.

Solution. Differentiating implicitly, $\dfrac{d}{d\theta}[\cos r + \cot\theta] = \dfrac{d}{d\theta}[e^{r\theta}]$ or $-\sin \overset{\curvearrowright}{r}[dr/d\theta] - \csc^2\theta = \overset{\curvearrowright}{e^{r\theta}}[[dr/d\theta](\theta) + (r)[1]]$ (notice that since r is a function of θ then we must use the Derivative Product Rule to differentiate $r\theta$) or $-\sin r(dr/d\theta) - \csc^2\theta = \theta e^{r\theta}(dr/d\theta) + re^{r\theta}$. Solving for $dr/d\theta$ we get $(-\sin r - \theta e^{r\theta})\dfrac{dr}{d\theta} = re^{r\theta} + \csc^2\theta$ or $\boxed{\dfrac{dr}{d\theta} = \dfrac{re^{r\theta} + \csc^2\theta}{-\sin r - \theta e^{r\theta}}}$. $\square$

Exercise 3.7.40

Exercise 3.7.40. Verify that the point $(\pi/4, \pi/2)$ is on the curve $x\sin 2y = y\cos 2x$ and find the equations of the lines that are **(a)** tangent and **(b)** normal to the curve at $(\pi/4, \pi/2)$.

Solution. First, with $(x, y) = (\pi/4, \pi/2)$ the equation $x\sin 2y = y\cos 2x$ becomes $\left(\dfrac{\pi}{4}\right)\sin\left(2\left(\dfrac{\pi}{2}\right)\right) \stackrel{?}{=} \left(\dfrac{\pi}{2}\right)\cos\left(2\left(\dfrac{\pi}{4}\right)\right)$ or $\left(\dfrac{\pi}{4}\right)\sin\pi \stackrel{?}{=} \left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{2}\right)$ or $\left(\dfrac{\pi}{4}\right)(0) \stackrel{?}{=} \left(\dfrac{\pi}{2}\right)(0)$ or $0 \stackrel{?}{=} 0$, which is true and so $(\pi/4, \pi/2)$ is on the curve $x\sin 2y = y\cos 2x$.

(a) We differentiate the equation implicitly to find dy/dx and we get $\dfrac{d}{dx}[x\sin 2y] = \dfrac{d}{dx}[y\cos 2x]$ or $[1](\sin 2y) + (x)[\cos\overset{\curvearrowright}{(2y)}[2dy/dx]] = [dy/dx](\cos 2x) + (y)[-\sin\overset{\curvearrowright}{(2x)}[2]]$ or $\sin 2y + 2x(\cos 2y)(dy/dx) = (\cos 2x)(dy/dx) - 2y\sin 2x$.

Exercise 3.7.40 (continued 1)

Exercise 3.7.40. Verify that the point $(\pi/4, \pi/2)$ is on the curve $x\sin 2y = y\cos 2x$ and find the equations of the lines that are **(a)** tangent and **(b)** normal to the curve at $(\pi/4, \pi/2)$.

Solution (continued).
$\ldots\sin 2y + 2x(\cos 2y)(dy/dx) = (\cos 2x)(dy/dx) - 2y\sin 2x$. Solving for dy/dx we get $(2x\cos 2y - \cos 2x)(dy/dx) = -2y\sin 2x - \sin 2y$ or $\dfrac{dy}{dx} = \dfrac{-2y\sin 2x - \sin 2y}{2x\cos 2y - \cos 2x}$. So at $(\pi/4, \pi/2)$ we have the slope of the tangent line as

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(\pi/4,\pi/2)} = \frac{-2(\pi/2)\sin 2(\pi/4) - \sin 2(\pi/2)}{2(\pi/4)\cos 2(\pi/2) - \cos 2(\pi/4)} = \frac{-\pi\sin(\pi/2) - \sin\pi}{(\pi/2)\cos\pi - \cos\pi/2} = \frac{-\pi(1) - (0)}{(\pi/2)(-1) - (0)} = \frac{-\pi}{-\pi/2} = 2.$$

So the equation of the line tangent to the curve at $(x_1, y_1) = (\pi/4, \pi/2)$ is, by the point-slope formula, $y - y_1 = m(x - x_1)$ or $y - (\pi/2) = (2)(x - (\pi/4))$ or $y = 2x - \pi/2 + \pi/2$ or $\boxed{y = 2x}$.

Exercise 3.7.40 (continued 2)

Exercise 3.7.40. Verify that the point $(\pi/4, \pi/2)$ is on the curve $x\sin 2y = y\cos 2x$ and find the equations of the lines that are **(a)** tangent and **(b)** normal to the curve at $(\pi/4, \pi/2)$.

Solution (continued). Since the slope of a tangent line to the curve at $(x_1, y_1) = (\pi/4, \pi/2)$ is 2, then the slope of a line perpendicular (or normal) to the curve at this point is the negative reciprocal of 2, and normal to the curve has slope $m = -1/2$. So by the point-slope formula, $y - y_1 = m(x - x_1)$, the normal line has formula $y - (\pi/2) = (-1/2)(x - (\pi/4))$ or $y = -x/2 + \pi/8 + \pi/2$ or $\boxed{y = -x/2 + 5\pi/8}$. $\square$

Exercise 3.7.44

Exercise 3.7.44. Normals Parallel to a Line.
Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

Solution. First we find dy/dx by differentiating implicitly to get: $\frac{d}{dx}[xy + 2x - y] = \frac{d}{dx}[0]$ or $[1](y) + (x)[dy/dx] + 2[1] - [dy/dx] = 0$ or $y + 2 + (x-1)dy/dx = 0$ or $\frac{dy}{dx} = \frac{-y-2}{x-1} = \frac{y+2}{1-x}$. So at a point (x_1, y_1) on the curve $xy + 2x - y = 0$, the slope of a normal line is $-1 \Big/ \left(\frac{dy}{dx}\right)\Big|_{(x_1,y_1)} = -\frac{1-x_1}{y_1+2}$. Now the slope of line $2x + y = 0$ is $m = -2$ ("$-A/B$," if you like). So we look for a point (x_1, y_1) such that $-\frac{1-x_1}{y_1+2} = -2$ or $1 - x_1 = 2(y_1 + 2) = 2y_1 + 4$ or $x_1 = -2y_1 - 3$.

Exercise 3.7.44 (continued)

Exercise 3.7.44. Normals Parallel to a Line.
Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

Solution (continued). So we look for a point (x_1, y_1) such that $\ldots x_1 = -2y_1 - 3$. Since (x_1, y_1) lies on the curve $xy + 2x - y = 0$, then we need $(-2y_1 - 3)y_1 + 2(-2y_1 - 3) - y_1 = 0$ or $-2y_1^2 - 3y_1 - 4y_1 - 6 - y_1 = 0$ or $-2y_1^2 - 8y_1 - 6 = 0$ or $y_1^2 + 4y_1 + 3 = 0$ or $(y_1 + 3)(y_1 + 1) = 0$. So we need $y_1 = -3$ **or** $y_1 = -1$ and then (from $xy + 2x - y = 0$) we have $x_1(-3) + 2x_1 - (-3) = 0$ and so $x_1 = 3$ **or** $x_1(-1) + 2x_1 - (-1) = 0$ and so $x_1 = -1$, respectively. That is, the desired normal lines occur at the points $(3, -3)$ and $(-1, -1)$. Since the slope of the normal line is $m = -2$ then by the point-slope formula the two normal lines are $y - (-3) = (-2)(x - 3)$ or $\boxed{y = -2x + 3}$ and $y - (-1) = (-2)(x - (-1))$ or $\boxed{y = -2x - 3}$. □

Exercise 3.7.22

Exercise 3.7.22. Find dy/dx and d^2y/dx^2 for y an implicit function of x given by the equation $x^{2/3} + y^{2/3} = 1$.

Solution. We differentiate implicitly to get $\frac{d}{dx}[x^{2/3} + y^{2/3}] = \frac{d}{dx}[1]$ or $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\left[\frac{dy}{dx}\right] = 0$ or $\frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$ or $\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$ or $\boxed{\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}}$.
Next we have, differentiating implicitly again, that
$\frac{d}{dx}\left[\frac{d}{dx}\right] = \frac{d}{dx}\left[-\frac{y^{1/3}}{x^{1/3}}\right]$ or

$$\frac{d^2y}{dx^2} = -\frac{[(1/3)y^{-2/3}[dy/dx]](x^{1/3}) - (y^{1/3})[(1/3)x^{-2/3}]}{(x^{1/3})^2} \text{ or } \ldots$$

Exercise 3.7.22 (continued)

Exercise 3.7.22. Find dy/dx and d^2y/dx^2 for y an implicit function of x given by the equation $x^{2/3} + y^{2/3} = 1$.

Solution. $\ldots \frac{d^2y}{dx^2} = -\frac{[(1/3)y^{-2/3}[dy/dx]](x^{1/3}) - (y^{1/3})[(1/3)x^{-2/3}]}{(x^{1/3})^2}$ or

$$\frac{d^2y}{dx^2} = -\frac{(1/3)x^{1/3}y^{-2/3}(dy/dx) - (1/3)x^{-2/3}y^{1/3}}{x^{2/3}} \text{ or}$$

$$\frac{d^2y}{dx^2} = -\frac{x^{1/3}y^{-2/3}(-y^{1/3}/x^{1/3}) - x^{-2/3}y^{1/3}}{3x^{2/3}} \text{ or}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{y^{-1/3} + x^{-2/3}y^{1/3}}{3x^{2/3}}}. \ \square$$

Exercise 3.7.48

Exercise 3.7.48. The Folium of Descartes.
(a) Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$. **(b)** At what point other than the origin does the folium have a horizontal tangent? **(c)** Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent

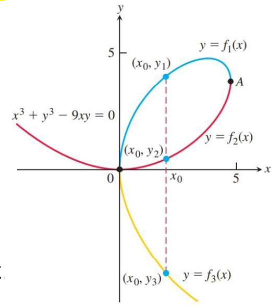


Figure 3.29

Solution. (a) Differentiating implicitly we have
$\frac{d}{dx}[x^3 + y^3 - 9xy] = \frac{d}{dx}[0]$ or

$$3x^2 + 3y^2\left[\frac{dy}{dx}\right] - 9\left[[1](y) + (x)\left[\frac{dy}{dx}\right]\right] = 0 \text{ or } (3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

...

Exercise 3.7.48 (continued 1)

(a) Find the slope of the folium of Descartes $x^3 + y^3 - 9xy = 0$ at the points $(4, 2)$ and $(2, 4)$.

Solution (continued). $\ldots(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$ or $\boxed{\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}}$ if $y^2 \neq 3x$. So the slope of the curve at $(4, 2)$ is
$\left.\frac{dy}{dx}\right|_{(x,y)=(4,2)} = \frac{9(2) - 3(4)^2}{3(2)^2 - 9(4)} = \frac{18 - 48}{12 - 36} = \frac{-30}{-24} = \boxed{\frac{5}{4}}$. The slope of the curve at $(4, 2)$ is $\left.\frac{dy}{dx}\right|_{(x,y)=(2,4)} = \frac{9(4) - 3(2)^2}{3(4)^2 - 9(2)} = \frac{36 - 12}{48 - 18} = \frac{24}{30} = \boxed{\frac{4}{5}}$. □

Exercise 3.7.48 (continued 2)

(b) At what point other than the origin does the folium have a horizontal tangent?

Solution (continued). (b) If the folium has a horizontal tangent at $(x, y) = (x_1, y_1)$ then
$\left.\frac{dy}{dx}\right|_{(x,y)=(x_1,y_1)} = \left.\frac{9y - 3x^2}{3y^2 - 9x}\right|_{(x,y)=(x_1,y_1)} = \frac{9y_1 - 3x_1^2}{3y_1^2 - 9x_1} = 0$. This implies $9y_1 - 3x_1^2 = 0$, or $y_1 = x_1^2/3$. Since (x_1, y_1) also lies on the folium $x^3 + y^3 - 9xy = 0$, then we must have $x_1^3 + (x_1^2/3)^3 - 9x_1(x_1^2/3) = 0$ or $x_1^3 + x_1^6/27 - 3x_1^3 = 0$ or $x_1^6/27 - 2x_1^3 = 0$ or $x_1^3(x_1^3/27 - 2) = 0$. So we need either $x_1 = 0$ or $x_1 = 3\sqrt[3]{2}$. When $x_1 = 0$ then $y_1 = 0$ and this is the horizontal tangent at the origin. When $x_1 = 3\sqrt[3]{2}$ we have $y_1 = (3\sqrt[3]{2})^2/3 = 9(2^{2/3})/3 = 3(\sqrt[3]{2^2}) = 3\sqrt[3]{4}$. So the other horizontal tangent occurs at $\boxed{(x_1, y_1) = (3\sqrt[3]{2}, 3\sqrt[3]{4})}$. □

Exercise 3.7.48 (continued 3)

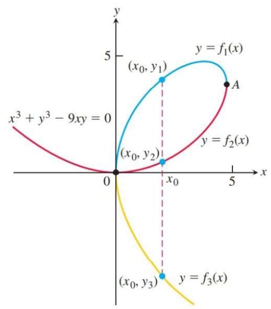
(c) Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent.

Solution (continued). (c) Since $\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$, then we look for a vertical tangent where the denominator is 0 and the numerator is not (see Dr. Bob's Infinite Limits Theorem in 2.6. Limits Involving Infinity; Asymptotes of Graphs and the definition of vertical tangent line in 3.1. Tangent Lines and the Derivative at a Point). So if the folium has a vertical tangent at (x_1, y_1) then $3y_1^2 - 9x_1 = 0$, or $x_1 = y_1^2/3$. Since (x_1, y_1) also lies on the folium $x^3 + y^3 - 9xy = 0$, then we must have $(y_1^2/3)^3 + y_1^3 - 9(y_1^2/3)y_1 = 0$ or $y_1^6/27 + y_1^3 - 3y_1^3 = 0$ or $y_1^6/27 - 2y_1^3 = 0$ or $y_1^3(y_1^3/27 - 2) = 0$. So we need either $y_1 = 0$ or $y_1 = 3\sqrt[3]{2}$. When $y_1 = 0$ then $x_1 = 0$ so that $(x_1, y_1) = (0, 0)$ and we see that the folium has both a horizontal tangent and a vertical tangent at the origin. When $y_1 = 3\sqrt[3]{2}$ we have $x_1 = (3\sqrt[3]{2})^2/3 = 3\sqrt[3]{2^2} = 3\sqrt[3]{4}$.

Exercise 3.7.48 (continued 4)

(c) Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent.

Solution (continued). We see from the graph that the folium has a vertical tangent at point A (where the x coordinate is near 5). We know that *if* the folium has a vertical tangent then it occurs at $(x_1, y_1) = (3\sqrt[3]{4}, 3\sqrt[3]{2})$ (or at the origin $(x_1, y_1) = (0, 0)$), so it must be that $\boxed{\text{point } A \text{ is } (3\sqrt[3]{4}, 3\sqrt[3]{2})}$. We have $3\sqrt[3]{4} \approx 4.76$, consistent with the graph. Also notice that the horizontal tangent is at $(3\sqrt[3]{2}, 3\sqrt[3]{4})$ by (b), reflecting the symmetry of the folium with respect to the line $y = x$. □



Exercise 3.7.50

Exercise 3.7.50. Power Rule for Rational Exponents.
Let p and q be integers with $q > 0$. If $y = x^{p/q}$, differentiate the equivalent equation $y^q = x^p$ implicitly and show that, for $y \neq 0$,

$$\frac{d}{dx}[x^{p/q}] = \frac{p}{q}x^{(p/q)-1}.$$

Solution. Now $y = x^{p/q}$ if and only if $y^q = (x^{p/q})^q = x^p$, so differentiating implicitly we have $\frac{d}{dx}[y^q] = \frac{d}{dx}[x^p]$ or (since $q > 0$ and $y \neq 0$) $qy^{q-1}\left[\frac{dy}{dx}\right] = px^{p-1}$ or $\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$ or, since $y = x^{p/q}$, $\frac{dy}{dx} = \frac{px^{p-1}}{q(x^{p/q})^{q-1}}$ or $\frac{dy}{dx} = \frac{px^{p-1}}{qx^{p-p/q}} = \frac{p}{q}x^{p-1-(p-p/q)}$ or $\frac{dy}{dx} = \frac{p}{q}x^{(p/q)-1}$, as claimed. □