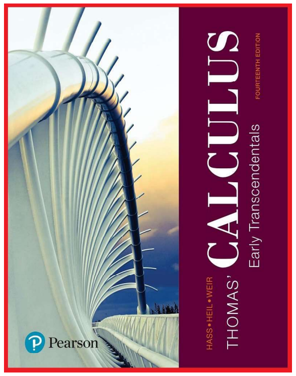
Calculus 1

Chapter 3. Derivatives

3.9. Inverse Trigonometric Functions—Examples and Proofs

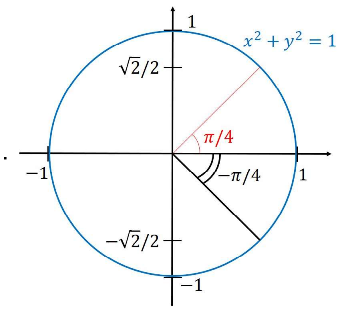


Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin\theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/6}$. $\square$

(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin\theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin\pi/4 = \sqrt{2}/2$. So we seek an angle θ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin\theta < 0$. We take $\boxed{\theta = -\pi/4}$.



$\square$

Exercise 3.9.4 (continued)

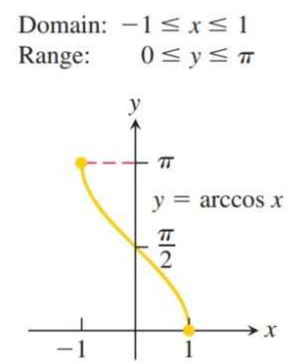
Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (c) With $\theta = \arcsin(\sqrt{3}/2)$, we need $\sin\theta = \sqrt{3}/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/3}$. $\square$

Exercise 3.9.14

Exercise 3.9.14. Find the limit: $\lim_{x\to -1^+} \cos^{-1}(x)$.

Solution. First, notice that -1 is a left endpoint of the domain of $\cos^{-1} x$. Based on the graph of $y = \cos^{-1} x$, we see (by Dr. Bob's Anthropomorphic Definition of Limit, a one-sided version) that as $x \to -1$ from the right (i.e., from the positive side) that the graph "tries to contain the point" $(-1, \pi)$. So $\lim_{x\to -1^+} \cos^{-1}(x) = \boxed{\pi}$. $\square$



Theorem 3.9.A

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}}\left[\frac{du}{dx}\right]$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$ or $\cos y \left[\frac{dy}{dx}\right] = 1$ or $\frac{dy}{dx} = \frac{1}{\cos y}$. Since we have restricted y to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1-(\sin y)^2} = \sqrt{1-x^2}$. Making this substitution we get $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$. The full theorem then follows from the Chain Rule. □

Exercise 3.9.24

Exercise 3.9.24. For dy/dt when $y = \sin^{-1}(1-t)$.

Solution. By Theorem 3.9.A (with $u(t) = 1-t$ and $du/dt = -1$), we have

$$\frac{dy}{dt} = \frac{d}{dt}[\sin^{-1}(1-t)] = \frac{1}{\sqrt{1-(1-t)^2}}[-1] = \boxed{\frac{-1}{\sqrt{2t-t^2}}}.$$

□

Theorem 3.9.B

Theorem 3.9.B. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\left[\frac{du}{dx}\right].$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation $\frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$ or $\sec^2 y \left[\frac{dy}{dx}\right] = 1$ or $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+(\tan y)^2} = \frac{1}{1+x^2}$. The full theorem then follows from the Chain Rule. □

Exercise 3.9.34

Exercise 3.9.34. Find dy/dx when $y = \tan^{-1}(\ln x)$.

Solution. By Theorem 3.9.B (with $u(x) = \ln x$ and $du/dx = 1/x$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\ln x)] = \frac{1}{1+(\ln x)^2}\left[\frac{1}{x}\right] = \boxed{\frac{1}{x(1+(\ln x)^2)}}.$$

□

Theorem 3.9.C

Theorem 3.9.C. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\left[\frac{du}{dx}\right]$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation $\frac{d}{dx}\left[\sec y\right] = \frac{d}{dx}\left[x\right]$ or $\sec y \tan y \left[\frac{dy}{dx}\right] = 1$ or $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$. We now need to express this last expression in terms of x. First, $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. Therefore we have

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Theorem 3.9.C (continued)

Proof (continued). ...

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Notice from the graph of $y = \sec^{-1} x$ above, that the slope of this function is positive wherever it is defined. So

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \begin{cases} +\frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

Notice that if $x > 1$ then $x = |x|$ and if $x < -1$ then $-x = |x|$. Therefore

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}.$$

The full theorem then follows from the Chain Rule. □

Exercise 3.9.40

Exercise 3.9.40. Find dy/dx when $y = \cot^{-1}(1/x) - \tan^{-1} x$.

Solution. By Table 3.1(3 and 4) (with $u(x) = 1/x = x^{-1}$ and $du/dx = -x^{-2} = -1/x^2$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(1/x) - \tan^{-1} x] = \frac{d}{dx}[\cot^{-1}(1/x)] - \frac{d}{dx}[\tan^{-1} x]$$

$$= \frac{-1}{1+(1/x)^2}\left[\frac{-1}{x^2}\right] - \frac{1}{1+x^2}$$

$$= \frac{1}{x^2(1+1/x^2)} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = \boxed{0}.$$

□

Exercise 3.9.44

Exercise 3.9.44. Find dy/dx at point $P(0, 1/2)$ when $\sin^{-1}(x+y) + \cos^{-1}(x-y) = 5\pi/6$.

Solution. Differentiating implicitly we have by Table 3.1(1 and 2) that
$\frac{d}{dx}[\sin^{-1}(x+y) + \cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right]$ or
$\frac{d}{dx}[\sin^{-1}(x+y)] + \frac{d}{dx}[\cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right]$ or

$$\frac{1}{\sqrt{1-(x+y)^2}}\left[1 + \frac{dy}{dx}\right] + \frac{-1}{\sqrt{1-(x-y)^2}}\left[1 - \frac{dy}{dx}\right] = 0 \text{ or}$$

$$\left(\frac{1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\right)\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}} \text{ or}$$

(getting a common denominator)

Exercise 3.9.44 (continued)

Exercise 3.9.44. Find dy/dx at point $P(0, 1/2)$ when $\sin^{-1}(x+y) + \cos^{-1}(x-y) = 5\pi/6$.

Solution (continued). $\ldots \left(\frac{\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}} \right) \frac{dy}{dx} = \frac{-\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}}$ or

$\left(\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}\right) \frac{dy}{dx} = -\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}$ or

$\frac{dy}{dx} = \frac{-\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}$. With $(x, y) = (0, 1/2)$ we have $\sqrt{1-(x \pm y)^2} = \sqrt{3/4} = \sqrt{3}/2$ and at $P(0, 1/2)$ we then have $\boxed{dy/dx|_{(x,y)=(0,1/2)} = 0}$. □

Exercise 3.9.60

Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution. Notice that

$$\frac{df}{dx} = \frac{d}{dx}\left[\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right] = \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}} \overset{\frown}{\frac{d}{dx}}\left[(x^2+1)^{-1/2}\right]$$

$$= \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}} \overset{\frown}{\left[\frac{-1}{2}(x^2+1)^{-3/2}\overset{\frown}{[2x]}\right]}$$

$$= \frac{1}{\sqrt{1-1/(x^2+1)}}(-x(x^2+1)^{-3/2}) = \frac{1}{\sqrt{((x^2+1)-1)/(x^2+1)}} \frac{-x}{(x^2+1)^{3/2}}$$

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2}} \frac{-x}{(x^2+1)\sqrt{x^2+1}} = \frac{-x}{|x|(x^2+1)}$$

Exercise 3.9.60 (continued 1)

Solution. Notice that

$$\frac{dg}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\frown}{\frac{d}{dx}}\left[\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\frown}{\left[\frac{-1}{x^2}\right]}$$

$$= \frac{-1}{(1+(1/x)^2)x^2} = \frac{-1}{x^2+1}.$$

So for $x > 0$, $f'(x) = g'(x)$. We will see in Corollary 4.2 (see Section 4.2. The Mean Value Theorem) that this implies $f(x) - g(x)$ is constant. We can evaluate f and g at some $x > 0$ to see what this constant is. With $x = 1$ we have

$f(1) = \sin^{-1}\frac{1}{\sqrt{(1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and

$g(1) = \tan^{-1}(1/(1)) = \tan^{-1}(1) = \pi/4$, so that the constant is 0 and so we must have $\boxed{f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}(1/x) = g(x) \text{ for } x > 0.}$

Exercise 3.9.60 (continued 2)

Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution (continued). For $x < 0$, $f'(x) = -g'(x)$ or $f'(x) + g'(x) = 0$. Again, by Corollary 4.2 (see Section 4.2. The Mean Value Theorem) this implies $f(x) + g(x)$ is constant. We can evaluate f and g at some $x < 0$ to see what this constant is. With $x = -1$ we have

$f(-1) = \sin^{-1}\frac{1}{\sqrt{(-1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and

$g(-1) = \tan^{-1}(1/(-1)) = \tan^{-1}(-1) = -\pi/4$, so that

$f(x) + g(x) = \pi/4 + (-\pi/4) = 0$ for $x < 0$, or

$\boxed{f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} = -\tan^{-1}(1/x) = -g(x) \text{ for } x < 0.}$ □