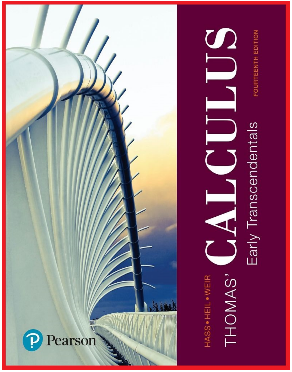
Calculus 1

Chapter 4. Applications of Derivatives

4.2. The Mean Value Theorem —Examples and Proofs



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Suppose that $y = f(x)$ is continuous at every point of $[a, b]$ and differentiable at every point of (a, b). If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Proof. Since f is continuous by hypothesis, f assumes an absolute maximum and minimum for $x \in [a, b]$ by Theorem 4.1 (The Extreme-Value Theorem for Continuous Functions). As seen in Section 4.1, these extrema occur only

1. at interior points where f' is zero
2. at interior points where f' does not exist
3. at the endpoints of the function's domain, a and b.

Since we have hypothesized that f is differentiable on (a, b), then Option 2 is not possible.

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Theorem 4.3 (continued)

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Proof (continued). In the event of Option 3 (extrema occur at the endpoints of the function's domain, a and b), the maximum and minimum occur at the endpoints a and b (where f is 0) and so f must be a constant of 0 throughout the interval. Therefore $f'(x) = 0$ for all $x \in (a, b)$, by Theorem 3.3.A (Derivative of a Constant Function), and the theorem holds. □

Exercise 4.2.60

Exercise 4.2.60. Rolle's Theorem Application.
(a) Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2. **(b)** Graph f and its derivative f' together. How is what you see related to Rolle's Theorem? **(c)** Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

Solution. (a) We take

$$\boxed{f(x) = (x+2)(x+1)x(x-1)(x-2)} = x(x^2-4)(x^2-1) = x^5-5x^3+4x$$

so that f has the desired zeros (and no others) and f is degree 5.

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(b) We have $f'(x) = 5x^4 - 15x^2 + 4$ so that f has critical points when $x^2 = \frac{-(-15)\pm\sqrt{(-15)^2-4(5)(4)}}{2(5)} = \frac{15\pm\sqrt{145}}{10}$; that is when $x = \pm\sqrt{\frac{15\pm\sqrt{145}}{10}}$, or $x = -\sqrt{\frac{15-\sqrt{145}}{10}} \approx -0.544$, $x = -\sqrt{\frac{15+\sqrt{145}}{10}} \approx -1.644$, $x = \sqrt{\frac{15-\sqrt{145}}{10}} \approx 1.644$, or $x = \sqrt{\frac{15+\sqrt{145}}{10}} \approx 0.544$.

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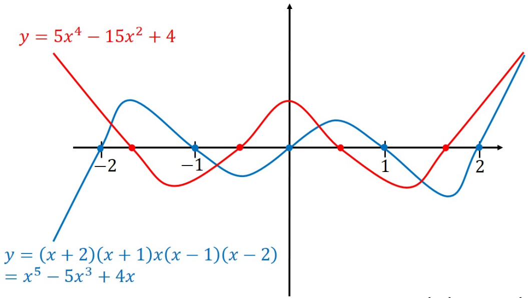
Exercise 4.2.60 (continued 1)

(b) Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

Solution (continued). The graph of $y = f(x)$ is:

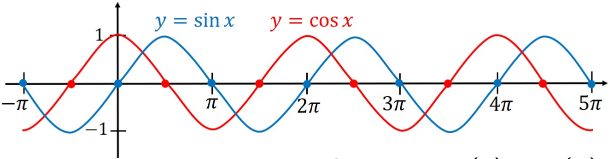


Notice that between each pair a and b for which $f(a) = f(b) = 0$ (i.e., the zeros of f, indicated by the five blue points) there is a c such that $f'(c) = 0$ (i.e., the zeros of f', indicated by the four red points), as required by Rolle's Theorem.

Exercise 4.2.60 (continued 2)

(c) Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon as f and f'?

Solution (continued). The graphs of $y = g(x) = \sin x$ and $y = g'(x) = \cos x$ are:



Notice that between each pair a and b for which $g(a) = g(b) = 0$ (i.e., the zeros of $g(x) = \sin x$, indicated by the blue points) there is a c such that $g'(c) = 0$ (i.e., the zeros of $g'(x) = \cos x$, indicated by the red points). So $\boxed{\text{yes}}$, the same behavior is the same as that of f and f' with respect to Rolle's Theorem. □

Theorem 4.4

Theorem 4.4. The Mean Value Theorem
Suppose that $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval (a, b). Then there is at least one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Consider the distinct points $A(a, f(a))$ and $B(b, f(b))$ on the graph of $y = f(x)$; see Figure 4.14. The secant line through these two points, from the point-slope form of a line, is $g(x) = \dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)$.

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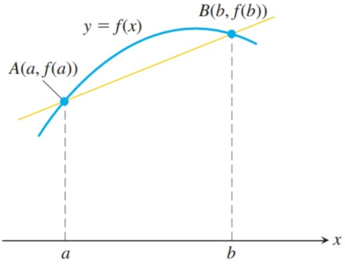

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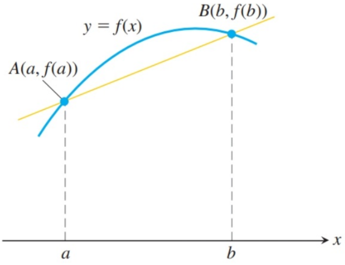

Figure 4.14

Theorem 4.4 (continued 1)

Proof (continued). Define the difference between the graphs of $y = f(x)$ and $y = g(x)$ as $h(x)$ so that

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right);$$

see Figure 4.15. The function h satisfies the hypotheses of Rolle's Theorem (Theorem 4.3; that's why we consider function h); h is continuous on $[a, b]$ and differentiable on (a, b). Also, $h(a) = h(b) = 0$. So, by Rolle's Theorem, $h'(c) = 0$ for some $c \in (a, b)$. We now show that c is the desired point for the conclusion of the Mean Value Theorem.

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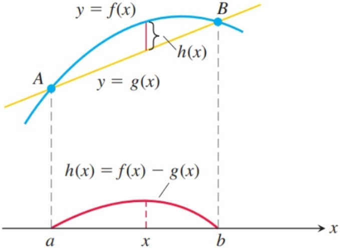

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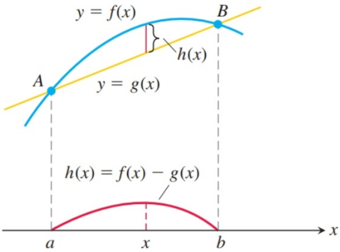

Figure 4.15

Theorem 4.4 (continued 2)

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Proof (continued). Since $h(x) = f(x) - \left(\dfrac{f(b) - f(a)}{b - a}(x - a) + f(a)\right)$, then $h'(x) = f'(x) - \dfrac{f(b) - f(a)}{b - a}$, and for $x = c$ we have

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

or

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

as claimed. □

Exercise 4.2.2

Exercise 4.2.2. Find the value of c that satisfies $\dfrac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the Mean Value Theorem for $f(x) = x^{2/3}$ on interval $[0,1]$.

Solution. We have $a = 0$, $b = 1$, $f(x) = x^{2/3}$, and $f'(x) = (2/3)x^{-1/3}$. So we seek $c \in (0,1)$ such that

$$f'(c) = (2/3)c^{-1/3} = \frac{f(b)-f(a)}{b-a} = \frac{(1)^{2/3}-(0)^{2/3}}{(1)-(0)} = 1.$$

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Exercise 4.2.52

Exercise 4.2.52. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

Solution. Introduce a Cartesian coordinate system where the truck is located at the origin when the 2 hours begin. We use units of hours on the horizontal t-axis and units of miles on the vertical y-axis. Let $f(t)$ represent the location of the truck at time t for the 2 hours under discussion (so that $f(0) = 0$ mi and $f(2) = 159$ mi). Then for physical reasons, f is differentiable for $t \in (0, 2)$ and continuous for $t \in [0, 2]$ so that the hypotheses of the Mean Value Theorem (Theorem 4.4) are satisfied.

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Prove. Consider the difference quotient for f as used in the alternative formula for derivative (see Exercise 3.2.24), $\dfrac{f(w) - f(x)}{w - x}$. By the hypothesis that $|f(w) - f(x)| \le |w - x|$, we have that the difference quotient satisfies $\left|\dfrac{f(w) - f(x)}{w - x}\right| \le 1$ for all $w \ne x$; that is $-1 \le \dfrac{f(w) - f(x)}{w - x} \le 1$ for all $w \ne x$. Now f is differentiable by hypothesis and by the alternative formula for derivative, $f'(x) = \lim_{w \to x} \dfrac{f(w) - f(x)}{w - x}$.

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Corollary 4.1

Corollary 4.1. Functions with Zero Derivatives Are Constant Functions.
If $f'(x) = 0$ at each point of an interval I, then $f(x) = k$ for all $x \in I$, where k is a constant.

Proof. Let x_1 and x_2 be any two points in (a, b) with $x_1 < x_2$. Then f is differentiable on $[x_1, x_2]$ and continuous on (x_1, x_2), so that we can apply the Mean Value Theorem to f on $[x_1, x_2]$. Therefore, there is $c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

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Corollary 4.2

Corollary 4.2. Functions with the Same Derivative Differ by a Constant

If $f'(x) = g'(x)$ at each point of an interval (a, b), then there exists a constant k such that $f(x) = g(x) + k$ for all $x \in (a, b)$.

Proof. Consider the function $h(x) = f(x) - g(x)$. Then we have $h'(x) = f'(x) - g'(x)$ and so $h'(x) = 0$ for all $x \in (a, b)$, by hypothesis. So $h(x)$ is constant on (a, b) by Corollary 4.1, say $h(x) = k$ for all $x \in (a, b)$. Therefore $f(x) - g(x) = k$ and $f(x) = g(x) + k$, as claimed. □

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Corollary 4.2. Functions with the Same Derivative Differ by a Constant
If $f'(x) = g'(x)$ at each point of an interval (a, b), then there exists a constant k such that $f(x) = g(x) + k$ for all $x \in (a, b)$.

Proof. Consider the function $h(x) = f(x) - g(x)$. Then we have $h'(x) = f'(x) - g'(x)$ and so $h'(x) = 0$ for all $x \in (a, b)$, by hypothesis. So $h(x)$ is constant on (a, b) by Corollary 4.1, say $h(x) = k$ for all $x \in (a, b)$. Therefore $f(x) - g(x) = k$ and $f(x) = g(x) + k$, as claimed. □

Exercise 4.2.40

Exercise 4.2.40. Find the function g with derivative $g'(x) = \dfrac{1}{x^2} + 2x$ whose graph passes through the point $P(-1, 1)$.

Solution. First, $g'(x) = x^{-2} + 2x$ and one function that has this as its derivative is $-x^{-1} + x^2$. We know by Corollary 4.2, "Functions with the Same Derivative Differ by a Constant," that any function with derivative $x^{-2} + 2x$ must be of the form $-x^{-1} + x^2 + k$ for some constant k. So we must have that g itself is of this form, $g(x) = -x^{-1} + x^2 + k$ for some k.

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Example 4.2.A

Example 4.2.A. Finding Velocity and Position from Acceleration. Suppose an object falls vertically in a gravitational field with constant acceleration of -9.8 m/sec^2. If the height at time t is given by $s(t)$ (so that $s''(t) = a(t) = -9.8$ m/sec^2), the initial height is $s(0) = s_0$ m, and the initial velocity is $s'(0) = v(0) = v_0$ m/sec, then find the velocity function $v(t)$ and the height function $s(t)$.

Solution. With $s(t)$ as position, $v(t)$ as velocity, and $a(t)$ as acceleration, we have $a(t) = v'(t)$ and $v(t) = s'(t)$. Since $a(t) = -9.8$ m/sec^2, then one function that has this as its derivative is $-9.8t$. We know by Corollary 4.2, "Functions with the Same Derivative Differ by a Constant," that any function with derivative -9.8 must be of the form $-9.8t + k_1$ for some constant k_1; in particular, $v(t) = -9.8t + k_1$ m/sec for some constant k_1. Since $v(0) = v_0$ then we have $v(0) = -9.8(0) + k_1 = v_0$ or $k_1 = v_0$. Therefore, $v(t) = -9.8t + v_0$.

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Example 4.2.A (continued)

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Solution (continued). Therefore, $v(t) = -9.8t + v_0$. One function that has $-9.8t + v_0$ as its derivative is $-4.9t^2 + v_0 t$, and by Corollary 4.2 any function with derivative $-9.8t + v_0$ is of the form $-4.9t^2 + v_0 t + k_2$ for come constant k_2; in particular, $s(t) = -4.9t^2 + v_0 t + k_2$ m for some constant k_2. Since $s(0) = s_0$ then we have $s(0) = -4.9(0)^2 + v_0(0) + k_2 = s_0$ or $k_2 = s_0$. Therefore, $\boxed{s(t) = -4.9t^2 + v_0 t + s_0 \text{ m}}$. □

Example 4.2.A (continued)

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Theorem 1.6.1/Theorem 4.2.A

Theorem 1.6.1/Theorem 4.2.A. Algebraic Properties of the Natural Logarithm

For any numbers $b > 0$ and $x > 0$ we have

1. $\ln bx = \ln b + \ln x$
2. $\ln \dfrac{b}{x} = \ln b - \ln x$
3. $\ln \dfrac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$.

Proof. First for (1). Notice that $\dfrac{d}{dx}[\ln bx] = \dfrac{1}{bx}\dfrac{d}{dx}[bx] = \dfrac{1}{bx}[b] = \dfrac{1}{x}$.

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Theorem 1.6.1/Theorem 4.2.A (continued 1)

For any numbers $b > 0$ and $x > 0$ we have

2. $\ln \dfrac{b}{x} = \ln b - \ln x$
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4. $\ln x^r = r \ln x$.

Proof (continued). Now for (2). We know by (1) that $\ln \dfrac{1}{x} + \ln x = \ln\left(\dfrac{1}{x}x\right) = \ln 1 = 0$. Therefore $\ln \dfrac{1}{x} = -\ln x$. Again by (1) we have $\ln \dfrac{b}{x} = \ln\left(b\dfrac{1}{x}\right) = \ln b + \ln \dfrac{1}{x} = \ln b - \ln x$. Notice that (3) follows from this with $b = 1$.

Theorem 1.6.1/Theorem 4.2.A (continued 1)

For any numbers $b > 0$ and $x > 0$ we have

2. $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^r = r \ln x$.

Proof (continued). Now for (2). We know by (1) that $\ln \frac{1}{x} + \ln x = \ln \left(\frac{1}{x} x\right) = \ln 1 = 0$. Therefore $\ln \frac{1}{x} = -\ln x$. Again by (1) we have $\ln \frac{b}{x} = \ln \left(b \frac{1}{x}\right) = \ln b + \ln \frac{1}{x} = \ln b - \ln x$. Notice that (3) follows from this with $b = 1$.

Theorem 1.6.1/Theorem 4.2.A (continued 2)

For any numbers $b > 0$ and $x > 0$ we have

4. $\ln x^r = r \ln x$.

Proof (continued). Now for part (4). We have by the Chain Rule (Theorem 3.2) and the General Power Rule for Derivatives (Theorem 3.3.C/3.8.D):

$$\frac{d}{dx}[\ln x^r] = \frac{1}{x^r}\frac{d}{dx}[x^r] = \frac{1}{x^r}\left[rx^{r-1}\right] = r\frac{1}{x} = r\frac{d}{dx}[\ln x] = \frac{d}{dx}[r \ln x].$$

As in the proof of (1), since $\ln xrn$ and $r \ln x$ have the same derivative, we have $\ln x^r = r \ln x + k_2$ for some k_2. With $x = 1$ we see that $k_2 = 0$ and we have $\ln x^r = r \ln x$. □

Theorem 1.6.1/Theorem 4.2.A (continued 2)

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Theorem 4.2.B(1)

Theorem 4.2.B. For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} e^{x_2} = e^{x_1 + x_2}$.

Proof. Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$. Then $\ln y_1 = \ln e^{x_1} = x_1$ and $\ln y_2 = \ln e^{x_2} = x_2$. So $x_1 + x_2 = \ln y_1 + \ln y_2 = \ln y_1 y_2$, and hence

$$e^{x_1 + x_2} = e^{\ln y_1 y_2} = y_1 y_2 = e^{x_1} e^{x_2},$$

as claimed. □

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