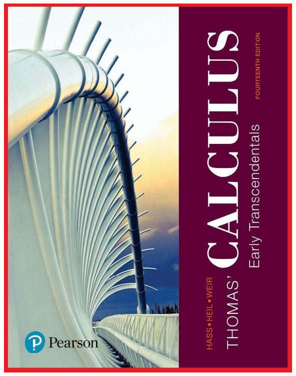
Calculus 1

Chapter 4. Applications of Derivatives

4.3. Monotone Functions and the First Derivative Test—Examples and Proofs



Corollary 4.3

Corollary 4.3. The First Derivative Test for Increasing and Decreasing.
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b)

If $f' > 0$ at each point of (a, b), then f increases on $[a, b]$.

If $f' < 0$ at each point of (a, b), then f decreases on $[a, b]$.

Proof. Suppose $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. The Mean Value Theorem (Theorem 4.4) applied to f on $[x_1, x_2]$ implies that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$ for some c between x_1 and x_2. Since $x_2 - x_1 > 0$, then $f(x_2) - f(x_1)$ and $f'(c)$ are of the same sign. Therefore $f(x_2) > f(x_1)$ if f' is positive on (a, b), and $f(x_2) < f(x_1)$ if f' is negative on (a, b). □

Exercise 4.3.28(a)

Exercise 4.3.28(a). Find the sets on which the function $g(x) = x^4 - 4x^3 + 4x^2$ is increasing and decreasing. Use the critical points of g to make a table of the sign of g' using test values from the intervals on which g' has the same sign.

Solution. We have

$$g'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2),$$

so the critical points of g are $x = 0$, $x = 1$, and $x = 2$ (where g' is 0). Since g' is a polynomial (and so is continuous by Theorem 2.5.A) then by the Intermediate Value Theorem (Theorem 2.11) the only way g' can change sign as x increases is for g' to take on the value 0. That is, g' has the same sign on the intervals $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$. So we use test values from these intervals to determine the sign of g' throughout these intervals.

Exercise 4.3.28(a) (continued 1)

Solution (continued). We have $g'(x) = 4x(x-1)(x-2)$. Consider:

interval	$(-\infty, 0)$	$(0, 1)$
test value k	-1	$1/2$
$g'(k)$	$4(-1)((-1)-1)((-1)-2)$	$4(1/2)((1/2)-1)((1/2)-2)$
$g'(x)$	$(-)(-)(-) = -$	$(+)(-)(-) = +$
$g(x)$	DEC	INC

interval	$(1, 2)$	$(2, \infty)$
test value k	$3/2$	4
$g'(k)$	$4(3/2)((3/2)-1)((3/2)-2)$	$4(4)((4)-1)((4)-2)$
$g'(x)$	$(+)(+)(-) = -$	$(+)(+)(+) = +$
$g(x)$	DEC	INC

So by Corollary 4.3 (The First Derivative Test for Increasing and Decreasing) g is **increasing on $[0, 1] \cup [2, \infty)$** and g is **decreasing on $(-\infty, 0] \cup [1, 2]$**.

Theorem 4.3.A

Theorem 4.3.A. First Derivative Test for Local Extrema.
Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a *local minimum* at c;
2. if f' changes from positive to negative at c, then f has a *local maximum* at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has *no local extremum* at c.

Proof. (1) Since the sign of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a, c)$ then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$ by Corollary 4.3 (The First Derivative Test for Increasing and Decreasing).

Theorem 4.3.A (continued 1)

Theorem 4.3.A. First Derivative Test for Local Extrema.

1. if f' changes from negative to positive at c, then f has a *local minimum* at c;
2. if f' changes from positive to negative at c, then f has a *local maximum* at c.

Proof (continued). If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$ by Corollary 4.3. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c. □

(2) Since the sign of f' changes from positive to negative at c, there are numbers a and b such that $a < c < b$, $f' > 0$ on (a, c), and $f' < 0$ on (c, b). If $x \in (a, c)$ then $f(c) > f(x)$ because $f' > 0$ implies that f is increasing on $[a, c]$ by Corollary 4.3. If $x \in (c, b)$, then $f(c) > f(x)$ because $f' < 0$ implies that f is decreasing on $[c, b]$ by Corollary 4.3. Therefore, $f(x) \leq f(c)$ for every $x \in (a, b)$. By definition, f has a local maximum at c. □

Theorem 4.3.A (continued 2)

Theorem 4.3.A. First Derivative Test for Local Extrema.

3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has *no local extremum* at c.

Proof (continued). (3) We show this in the case that f' is positive on both sides of c, the case that f' is negative on both sides of c being similar. Then there are numbers a and b such that $a < c < b$, $f' > 0$ on (a, c), and $f' > 0$ on (c, b). If $x \in (a, c)$ then $f(c) > f(x)$ because $f' > 0$ implies that f is increasing on $[a, c]$ by Corollary 4.3. If $y \in (c, b)$, then $f(c) < f(y)$ because $f' > 0$ implies that f is increasing on $[c, b]$ by Corollary 4.3. Therefore, $f(x) \leq f(c) \leq f(y)$ for every $x \in (a, c)$ and every $y \in (c, b)$. By definition, f has a neither a local maximum nor a local minimum at c. □

Exercise 4.3.28(b)

Exercise 4.3.28(b). Identify the local and absolute extreme values, if any, of $g(x) = x^4 - 4x^3 + 4x^2$.

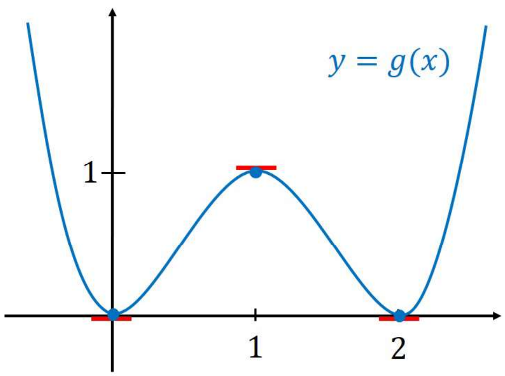
Solution. From part (a) above, we have

interval	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$g'(x)$	$-$	$+$	$-$	$+$
$g(x)$	DEC	INC	DEC	INC

So by Theorem 4.3.A (First Derivative Test for Local Extrema), g has a
local minimum at $x = 0$ of $g(0) = (0)^4 - 4(0)^3 + 4(0)^2 = 0$,
local minimum at $x = 2$ of $g(2) = (2)^4 - 4(2)^3 + 4(2)^2 = 0$, and
g has a local maximum at $x = 1$ of $g(1) = (1)^4 - 4(1)^3 + 4(1)^2 = 1$.

Exercise 4.3.28(b) (continued)

Solution (continued). Plotting the points of local extreme values, the critical points, and the increasing/decreasing information, we can get a good idea of the shape of the graph of $y = g(x) = x^4 - 4x^3 + 4x^2$:



□

Exercise 4.3.14

Exercise 4.3.14. Consider function f defined on $[a, b] = [0, 2\pi]$ with derivative $f'(x) = (\sin x + \cos x)(\sin x - \cos x)$. **(a)** What are the critical points? **(b)** On what sets is f increasing or decreasing? **(c)** At what points, if any, does f assume local maximum and minimum values?

Solution. First, $f'(x) = \sin^2 x - \cos^2 x$ and so $f'(x) = -(\cos^2 x - \sin^2 x) = -\cos 2x$ by the double angle formula.

(a) For the critical points, we consider $f'(x) = -\cos 2x = 0$ on $[0, 2\pi]$, or $\cos 2x = 0$ on $[0, 2\pi]$. So the critical points in $[0, 2\pi]$ are $\boxed{x = \pi/4,\ x = 3\pi/4,\ x = 5\pi/4, \text{ and } x = 7\pi/4}$.

(b) As in Exercise 4.3.28(a) above, we use the critical points in $[0, 2\pi]$ to determine the intervals in $[0, 2\pi]$ on which the sign of f' is constant: $[0, \pi/4)$, $(\pi/4, 3\pi/4)$, $(3\pi/4, 5\pi/4)$, $(5\pi/4, 7\pi/4)$, and $(7\pi/4, 2\pi]$.

Exercise 4.3.14 (continued 1)

Solution (continued). We have $f'(x) = -\cos 2x$, so:

interval	$[0, \pi/4)$	$(\pi/4, 3\pi/4)$	$(3\pi/4, 5\pi/4)$
test value k	0	$\pi/2$	π
$f'(k)$	$-\cos(2(0)) = -1$	$-\cos(2(\pi/2)) = 1$	$-\cos(2(\pi)) = -1$
$f'(x)$	$-$	$+$	$-$
$f(x)$	DEC	INC	DEC

interval	$(5\pi/4, 7\pi/4)$	$(7\pi/4, 2\pi]$
test value k	$3\pi/2$	2π
$f'(k)$	$-\cos(2(3\pi/2)) = 1$	$-\cos(2(2\pi)) = -1$
$f'(x)$	$+$	$-$
$f(x)$	INC	DEC

So f is $\boxed{\text{increasing on } [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]}$, and f is $\boxed{\text{decreasing on } [0, \pi/4] \cup [3\pi/4, 5\pi/4] \cup [7\pi/4, 2\pi]}$.

Exercise 4.3.14 (continued 2)

Exercise 4.3.14. Consider $f(x) = (\sin x + \cos x)(\sin x - \cos x)$ on $[a, b] = [0, 2\pi]$. **(c)** At what points, if any, does f assume local maximum and minimum values?

Solution (continued). ...So f is $\boxed{\text{increasing on } [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]}$, and f is $\boxed{\text{decreasing on } [0, \pi/4] \cup [3\pi/4, 5\pi/4] \cup [7\pi/4, 2\pi]}$.

(c) By Theorem 4.3.A ("First Derivative Test for Local Extrema"), f has a $\boxed{\text{local maximum at } x = 3\pi/4 \text{ and } x = 7\pi/4}$, and f has a $\boxed{\text{local minimum at } x = \pi/4 \text{ and } x = 5\pi/4}$. Since f decreases on $[0, \pi/4]$ then f has a $\boxed{\text{local maximum at } x = 0}$, and since f is decreasing on $[7\pi/4, 2\pi]$ then f has a $\boxed{\text{local minimum at } x = 2\pi}$. □

Exercise 4.3.38

Exercise 4.3.38. (a) Find the sets on which the function $g(x) = x^{2/3}(x+5)$ is increasing and decreasing. Use the critical points of g to make a table of the sign of g' using test values from the intervals on which g' has the same sign. **(b)** Identify the local and absolute extreme values of g, if any.

Solution. First,

$$g'(x) = [(2/3)x^{-1/3}](x+5) + (x^{2/3})[1] = \frac{2(x+5)}{3x^{1/3}} + \frac{3x}{3x^{1/3}} = \frac{5x+10}{3x^{1/3}}.$$

(a) For the critical points, since $g'(x) = \dfrac{5x+10}{3x^{1/3}}$ then $x = -2$ is a critical point since $g'(-2) = 0$ and $x = 0$ is a critical point since g' is not defined at $x = 0$. As in Exercise 4.3.28(a) above, we see that the sign of g' is the same throughout each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$. So we consider: ...

Exercise 4.3.38 (continued 1)

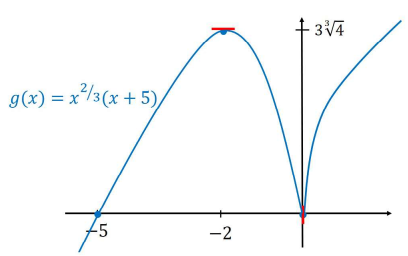
Solution (continued). So we consider:

interval	$(-\infty, -2)$	$(-2, 0)$	$(0, \infty)$
test value k	-3	-1	1
$g'(k)$	$\frac{5(-3)+10}{3(-3)^{1/3}}$ $= (-5/3)(1/(-3)^{1/3})$	$\frac{5(-1)+10}{3(-1)^{1/3}}$ $= (5/3)(1/(-1)^{1/3})$	$\frac{5(1)+10}{3(1)^{1/3}}$ $= 5$
$g'(x)$	$+$	$-$	$+$
$g(x)$	INC	DEC	INC

So g is **increasing on $(-\infty, -2] \cup [0, \infty)$**, and g is **decreasing on $[-2, 0]$**.

Exercise 4.3.38 (continued 2)

Solution (continued). (b) By Theorem 4.3.A ("First Derivative Test for Local Extrema"), g has a **local maximum at $x = -2$** (of $g(-2) = (-2)^{2/3}((-2)+5) = 3\sqrt[3]{4}$), and g has a **local minimum at $x = 0$** (of $g(0) = (0)^{2/3}((0)+5) = 0$). Now $g(x) = x^{2/3}(x+5)$ can be made arbitrarily large and positive by making x large and positive, and $g(x) = x^{2/3}(x+5)$ can be made arbitrarily large and negative by making x large and negative. So g has **no absolute extrema**. □



□

Exercise 4.3.44

Exercise 4.3.44. (a) Find the open intervals on which the function $f(x) = x^2 \ln x$ is increasing and decreasing. Use the critical points of f to make a table of the sign of f' using test values from the intervals on which f' has the same sign. **(b)** Identify the local and absolute extreme values of f, if any.

Solution. First, notice that the domain of f is $(0, \infty)$ and $f'(x) = [2x](\ln x) + (x^2)[1/x] = x(1 + 2\ln x)$.

(a) For the critical point(s), since $f'(x) = x(1 + 2\ln x)$ we see that we have $f'(x) = 0$ when $\ln x = -1/2$ or $e^{\ln x} = e^{-1/2}$ or $x = e^{-1/2}$ (notice that $x = 0$ is not in the domain of f). So we use the critical point to break the domain of f into open intervals and consider...

Exercise 4.3.44 (continued)

Solution (continued).

interval	$(0, e^{-1/2})$	$(e^{-1/2}, \infty)$
test value k	$e^{-3/4}$	1
$f'(k)$	$(e^{-3/4})(1 + 2\ln(e^{-3/4}))$ $e^{-3/4}(1 - 3/2)$	$(1)(1 + 2\ln(1))$ 1
$f'(x)$	$-$	$+$
$f(x)$	DEC	INC

So f is $\boxed{\text{increasing on } (e^{-1/2}, \infty)}$, and f is $\boxed{\text{decreasing on } (0, e^{-1/2})}$.

(b) By Theorem 4.3.A ("First Derivative Test for Local Extrema"), f has a $\boxed{\text{local minimum at } x = e^{-1/2}}$ (of $f(e^{-1/2}) = (e^{-1/2})^2 \ln e^{-1/2} = e^{-1}(-1/2) = -1/(2e)$). □

Exercise 4.3.58

Exercise 4.3.58. Consider $g(x) = \dfrac{x^2}{4 - x^2}$ on $(-2, 1]$. **(a)** Identify the local extreme values of g and say where they occur. **(b)** Which of the extreme values, if any, are absolute?

Solution. Notice that g is a rational function and, by Dr. Bob's Infinite Limits Theorem, g has a vertical asymptote at $x = -2$. Now $g'(x) = \dfrac{[2x](4 - x^2) - (x^2)[-2x]}{(4 - x^2)^2} = \dfrac{8x}{(4 - x^2)^2}$, and $x = 0$ is a critical point of g since $g'(0) = 0$ (notice that there are no other critical points of g).

(a) We partition the interval $(-2, 1]$ by removing the critical point $x = 0$ and consider: ...

Exercise 4.3.58 (continued 1)

Exercise 4.3.58. Consider $g(x) = \dfrac{x^2}{4 - x^2}$ on $(-2, 1]$. **(a)** Identify the local extreme values of g and say where they occur. **(b)** Which of the extreme values, if any, are absolute?

Solution (continued). (a) We partition the interval $(-2, 1]$ by removing the critical point $x = 0$ and consider: ...

interval	$(-2, 0)$	$(0, 1]$
test value k	-1	1
$g'(k)$	$(8(-1))/(4 - (-1)^2)^2 = -8/9$	$(8(1))/(4 - (1)^2)^2 = 8/9$
$g'(x)$	$-$	$+$
$g(x)$	DEC	INC

By Theorem 4.3.A ("First Derivative Test for Local Extrema"), g has a $\boxed{\text{local minimum at } x = 0 \text{ of } g(0) = 0}$. Since g is increasing on $[0, 1]$, then g has a $\boxed{\text{local maximum at } x = 1 \text{ of } g(1) = (1)^2/(4 - (1)^2) = 1/3}$.

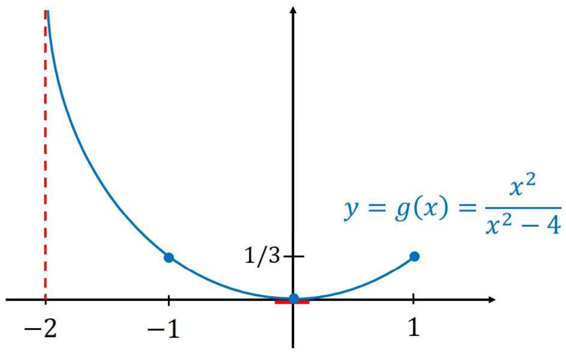
Exercise 4.3.58 (continued 2)

Exercise 4.3.58. Consider $g(x) = \dfrac{x^2}{4 - x^2}$ on $(-2, 1]$. **(a)** Identify the local extreme values of g and say where they occur. **(b)** Which of the extreme values, if any, are absolute?

Solution (continued). (b) Notice that since g has a vertical asymptote at $x = -2$ and g is decreasing on $(-2, 0)$, then we must have $\lim_{x \to -2^+} g(x) = \infty$ and so $\boxed{g \text{ has no absolute maximum}}$. Since g has a local minimum at $x = 0$ of $g(0) = 0$, g is decreasing on $(-2, 0]$, and g is increasing on $[0, 1]$, then g has an $\boxed{\text{absolute minimum at } x = 0 \text{ of } g(0) = 0}$.

Exercise 4.3.58 (continued 3)

Solution (continued). Plotting the points of local extreme values, the critical point, the vertical asymptote, and the increasing/decreasing information, we can get a good idea of the shape of the graph of $y = g(x) = \dfrac{x^2}{4 - x^2}$ on $(-2, 1]$:

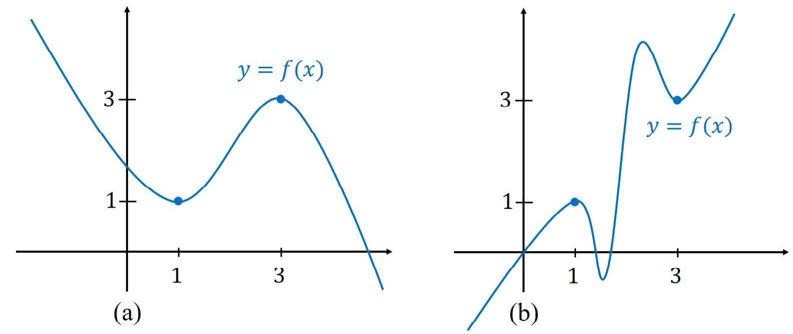


□

Exercise 4.3.72

Exercise 4.3.72. Sketch the graph of a differentiable function $y = f(x)$ that has **(a)** a local minimum at $(1, 1)$ and a local maximum at $(3, 3)$; **(b)** a local maximum at $(1, 1)$ and a local minimum at $(3, 3)$; **(c)** local maxima at $(1, 1)$ and $(3, 3)$; **(d)** local minima at $(1, 1)$ and $(3, 3)$.

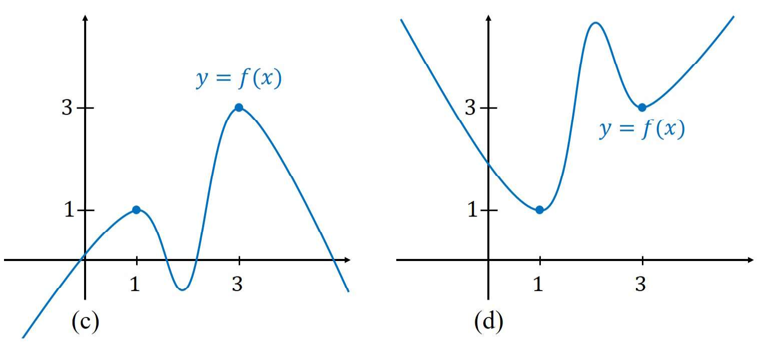
Solution. We try to make $y = f(x)$ simple by minimizing the number of critical points.



Exercise 4.3.72 (continued)

Exercise 4.3.72. Sketch the graph of a differentiable function $y = f(x)$ that has **(c)** local maxima at $(1, 1)$ and $(3, 3)$; **(d)** local minima at $(1, 1)$ and $(3, 3)$.

Solution(continued). We try to make $y = f(x)$ simple by minimizing the number of critical points.



□

Exercise 4.3.80

Exercise 4.3.80. (a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$. **(b)** Using part (a), show that $\ln x < x$ if $x > 1$.

Solution. Notice that the domain of f is $(0, \infty)$. We have $f'(x) = 1 - 1/x = (x - 1)/x$, so $x = 1$ is a critical point of f since $f'(1) = 0$.

(a) We partition the domain $(0, \infty)$ by removing the critical point $x = 1$ and consider:

interval	$(0, 1)$	$(1, \infty)$
test value k	$1/2$	2
$g'(k)$	$((1/2) - 1)/(1/2) = -1$	$((2) - 1)/(2) = 1/2$
$g'(x)$	$-$	$+$
$g(x)$	DEC	INC

So f is increasing on $[1, \infty)$ (in particular, for $x > 1$), as claimed; we are using a version of Corollary 4.3, "The First Derivative Test for Increasing and Decreasing" here. □

Exercise 4.3.80 (continued)

Exercise 4.3.80. (a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.
(b) Using part (a), show that $\ln x < x$ if $x > 1$.

Solution (continued). So f is increasing on $[1, \infty)$ (in particular, for $x > 1$), as claimed; we are using a version of Corollary 4.3, "The First Derivative Test for Increasing and Decreasing" here. □

(b) Since $f(1) = (1) - \ln(1) = 1 - 0 = 1$ and $f(x) = x - \ln x$ is increasing on $[1, \infty)$, then we have $f(x) = x - \ln x \geq 1$ for $x > 1$. That is, $x \geq \ln x + 1 > \ln x$ for $x > 1$, as claimed. □