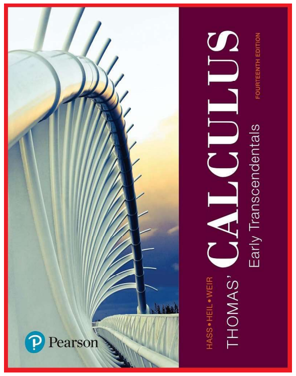
Calculus 1

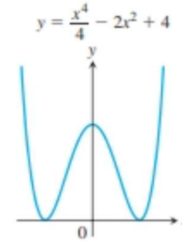
Chapter 4. Applications of Derivatives

4.4. Concavity and Curve Sketching—Examples and Proofs



Exercise 4.4.2

Exercise 4.4.2. Consider $f(x) = x^4/4 - 2x^2 + 4$. Identify the inflection points and local maxima and minima of f and identify the intervals on which the function is concave up and concave down.



Solution. First, $f'(x) = x^3 - 4x = x(x^2 - 4) = x(x+2)(x-2)$ and we see that -2, 0, and 2 are critical points since f' is 0 at these points. Next, $f''(x) = 3x^2 - 4$ so that $x = \pm\sqrt{4/3} = \pm 2/\sqrt{3}$ are *potential* points of inflection.

Exercise 4.4.2 (continued 1)

Solution (continued). As in the previous section, since $f''(x) = 3x^2 - 4$ is a polynomial (and so is continuous by Theorem 2.5.A) then by the Intermediate Value Theorem (Theorem 2.11) the only way f'' can change sign as x increases is for f'' to take on the value 0. That is, f'' has the same sign on the intervals $(-\infty, -2/\sqrt{3})$, $(-2/\sqrt{3}, 2/\sqrt{3})$, and $(2/\sqrt{3}, \infty)$. So we use test values from these intervals to determine the sign of f'' throughout these intervals.

interval	$(-\infty, -2/\sqrt{3})$	$(-2/\sqrt{3}, 2/\sqrt{3})$	$(2/\sqrt{3}, \infty)$
test value k	-2	0	2
$f''(k)$	$3(-2)^2 - 4 = 8$	$3(0)^2 - 4 = -4$	$3(2)^2 - 4 = 8$
$f''(x)$	+	−	+
$f(x)$	CU	CD	CU

Here, the concavity is given by the Second Derivative Test for Concavity (Theorem 4.4.A).

Exercise 4.4.2 (continued 2)

Solution (continued). ...

interval	$(-\infty, -2/\sqrt{3})$	$(-2/\sqrt{3}, 2/\sqrt{3})$	$(2/\sqrt{3}, \infty)$
$f(x)$	CU	CD	CU

So f *does* in fact change concavity at both $x = -2/\sqrt{3}$ and $x = 2/\sqrt{3}$. Notice $f(\pm 2/\sqrt{3}) = (\pm 2/\sqrt{3})^4/4 - 2(\pm 2/\sqrt{3})^2 + 4 = 4/9 - 8/3 + 4 = 4/9 - 24/9 + 36/9 = 16/9$. So by definition, the $\boxed{\text{inflection points are } (-2/\sqrt{3}, 16/9) \text{ and } (2/\sqrt{3}, 16/9))}$. f is $\boxed{\text{CU on } (-\infty, -2/\sqrt{3}) \cup (2/\sqrt{3}, \infty)}$ and f is $\boxed{\text{CD on } (-2/\sqrt{3}, 2/\sqrt{3})}$. We are given the graph of f, so we see that it has a $\boxed{\text{local maximum of } f(0) = (0)^4/4 - 2(0)^2 + 4 = 4}$ and a $\boxed{\text{local minimum of } f(-2) = f(2) = (2)^4/4 - 2(2)^2 + 4 = 0}$. □

Theorem 4.5

Theorem 4.5. Second Derivative Test for Local Extrema.
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof. (1) If $f'' < 0$, then $f'' < 0$ on some open interval I containing the point c, since f'' is continuous (by Exercise 2.5.70). Therefore by Corollary 4.3 ("The First Derivative Test for Increasing and Decreasing"), f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative as x increases through the value c, and so f has a local maximum at $x = c$ by Theorem 4.3.A(2), "First Derivative Test for Local Extrema," as claimed.

Theorem 4.5 (continued 1)

Theorem 4.5. Second Derivative Test for Local Extrema.
Suppose $f''$is continuous on an open interval that contains $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof. (2) If $f'' > 0$, then $f'' > 0$ on some open interval I containing the point c, since f'' is continuous (by Exercise 2.5.70). Therefore by Corollary 4.3 ("The First Derivative Test for Increasing and Decreasing"), f' is increasing on I. Since $f'(c) = 0$, the sign of f' changes from negative to positive as x increases through the value c, and so f has a local minimum at $x = c$ by Theorem 4.3.A(2), "First Derivative Test for Local Extrema," as claimed.

Theorem 4.5 (continued 2)

Theorem 4.5. Second Derivative Test for Local Extrema.
Suppose $f''$is continuous on an open interval that contains $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Proof. (3) We establish by this by giving examples. Consider $f_1(x) = x^4$, $f_2(x) = -x^4$, and $f_3(x) = x^3$. We have $f_1'(0) = f_2'(0) = f_3'(0) = 0$ (so we take $c = 0$), and $f_1''(0) = f_2''(0) = f_3''(0) = 0$. But $f_1(x) = x^4$ has a local minimum at $x = 0$, $f_2(x) = -x^4$ has a local maximum at $x = 0$, and $f_3(x) = x^3$ has neither a maximum nor a minimum at $x = 0$. So, as claimed, the test fails (is "inconclusive"). □

Exercise 4.4.12

Exercise 4.4.12. Consider $y = f(x) = x(6 - 2x)^2$. Identity the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$.

Solution. First, $f'(x) = [1](6 - 2x)^2 + (x)[2(6 - 2x)[-2]] = (6 - 2x)((6 - 2x) - 4x) = (6 - 2x)(6 - 6x)$ so that $x = 1$ and $x = 3$ are critical points since f' is 0 at these points. Next $f''(x) = [-2](6-6x)+(6-2x)[-6] = -12+12x-36+12x = -48+24x$, so $x = 2$ is a *potential* point of inflection. As above, since f'' is continuous then we test the sign of f'' as:

interval	$(-\infty, 2)$	$(2, \infty)$
test value k	1	3
$f''(k)$	$-48 + 24(1) = -24$	$-48 + 24(3) = 24$
$f''(x)$	$-$	$+$
$f(x)$	CD	CU

Exercise 4.4.12 (continued 1)

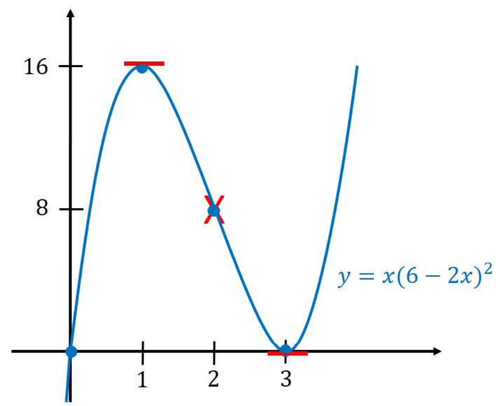
Solution (continued). So f *does* in fact change concavity at $x = 2$. Notice $f(2) = (2)(6 - 2(2))^2 = 8$ so the $\boxed{\text{point of inflection is } (2, 8)}$. We used the critical points as test values above, so we see by the Second Derivative Test for Local Extrema (Theorem 4.5) that f has a local maximum at $x = 1$ of $f(1) = (1)(6 - 2(1))^2 = 16$ and f has a local minimum at $x = 3$ of $f(3) = (3)(6 - 2(3))^2 = 0$. The coordinates of the $\boxed{\text{local maximum point is } (1, 16)}$ and the coordinates of the $\boxed{\text{local minimum point is } (3, 0)}$.

To graph $y = f(x)$, we plot each extreme point and the point of inflection. We use little horizontal hash marks "—" through the extreme points (since tangent lines are horizontal there) and we use a "**X**" to indicate a point of inflection. We also plot the x-intercepts $(0, 0)$ and $(3, 0)$, and the y-intercept $(0, 0)$. Finally, we flesh out the graph in a way that reflects the known concavity.

Exercise 4.4.12 (continued 2)

Exercise 4.4.12. Consider $y = f(x) = x(6 - 2x)^2$. Identity the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$.

Solution (continued). We then have:



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Exercise 4.4.104

Exercise 4.4.104. Sketch a smooth connected curve $y = f(x)$ with: $f(-2) = 8$, $f(0) = 4$, $f(2) = 0$, $f'(x) > 0$ for $|x| > 2$, $f'(2) = f'(-2) = 0$, $f'(x) < 0$ for $|x| < 2$, $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$. Indicate points where f' is 0 with horizontal hash marks and indicate points of inflection with X's.

Solution. Since $f'(x) > 0$ for $|x| > 2$ and $f'(x) < 0$ for $|x| < 2$, then by The First Derivative Test for Increasing and Decreasing (Corollary 4.3) f is INC on $(-\infty, -2) \cup (2, \infty)$ and f is DEC on $(-2, 2)$. Since $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, the by the Second Derivative Test for Concavity (Theorem 4.4.A) f is CU on $(0, \infty)$ and f is CD on $(-\infty, 0)$. We combine this information in a table:

interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	+	−	−	+
$f''(x)$	−	−	+	+
$f(x)$	INC, CD	DEC, CD	DEC, CU	INC, CU

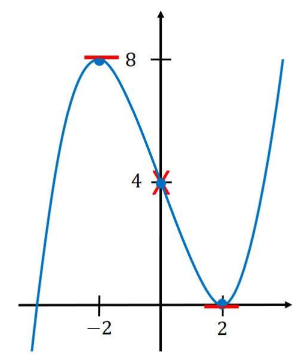
Notice that $(0, f(0)) = (0, 4)$ is a point of inflection.

Exercise 4.4.104 (continued)

Solution (continued). ...

interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 2)$	$(2, \infty)$
$f(x)$	INC, CD	DEC, CD	DEC, CU	INC, CU

Plotting the points $f(-2) = 8$, $f(0) = 4$, $f(2) = 0$, and using the INC/DEC and CU/CD information, along with the fact that f is "smooth" gives:



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Exercise 4.4.42

Exercise 4.4.42. Consider $y = f(x) = \sqrt[3]{x^3+1}$. Identity the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$. Indicate points where f' is 0 with horizontal hash marks and indicate points of inflection with X's.

Solution. First, $f(x) = (x^3+1)^{1/3}$ and so

$$f'(x) = (1/3)(x^3+1)^{-2/3}\overset{\curvearrowright}{[3x^2]} = x^2(x^3+1)^{-2/3} = \frac{x^2}{(x^3+1)^{2/3}}, \text{ so } x = 0$$

is a critical point since $f'(0) = 0$ and $x = -1$ is a critical point since $x = -1$ is in the domain of f but f' is undefined at $x = -1$. Next

$$f''(x) = [2x]((x^3+1)^{-2/3}) + (x^2)[(-2/3)(x^3+1)^{-5/3}\overset{\curvearrowright}{[3x^2]}]$$

$$= \frac{2x}{(x^3+1)^{2/3}} - \frac{2x^4}{(x^3+1)^{5/3}} = \frac{2x(x^3+1)-2x^4}{(x^3+1)^{5/3}} = \frac{2x}{(x^3+1)^{5/3}},$$

so f has a potential point of inflection at $x = 0$ and at $x = -1$ (notice that f'' is undefined at $x = -1$, but we could show that $y = f(x)$ has a vertical tangent at $x = -1$).

Exercise 4.4.42 (continued 1)

Solution (continued). We find the signs of $f'(x) = x^2/(x^3+1)^{2/3}$ and $f''(x) = 2x/(x^3+1)^{5/3}$ over the appropriate intervals:

interval	$(-\infty,-1)$	$(-1,0)$	$(0,\infty)$
test value k	-2	$-1/2$	1
$f'(k)$	$(-2)^2/((-2)^3+1)^{2/3}$	$(-1/2)^2/((-1/2)^3+1)^{2/3}$	$(1)^2/((1)^3+1)^{2/3}$
$f'(x)$	$(+)/(+) = +$	$(+)/(+) = +$	$(+)/(+) = +$
$f''(k)$	$2(-2)/((-2)^3+1)^{5/3}$	$2(-1/2)/((-1/2)^3+1)^{5/3}$	$2(1)/((1)^3+1)^{5/3}$
$f''(x)$	$(-)/(-) = +$	$(-)/(+) = -$	$(+)/(+) = +$
$f(x)$	INC, CU	INC, CD	INC, CU

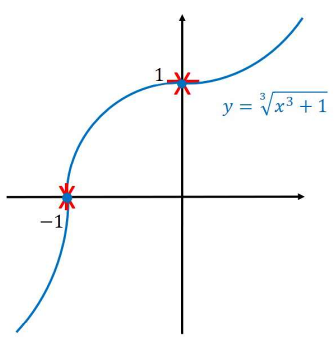
Since f is always increasing then it has no local maximum nor local minimum (by the First Derivative Test for Local Extrema, Theorem 4.3.A(3)). Notice that f changes concavity at $x = -1$ and $x = 0$, so the points of inflection are $(-1, f(-1)) = (-1, 0)$ and $(0, f(0)) = (0, 1)$.

Exercise 4.4.42 (continued 2)

Solution (continued). Since $f'(0) = 0$, f' is undefined at $x = -1$, $f(-1) = 0$, $f(0) = 1$, and

interval	$(-\infty,-1)$	$(-1,0)$	$(0,\infty)$
$f(x)$	INC, CU	INC, CD	INC, CU

then the graph of $y = f(x) = \sqrt[3]{x^3+1}$ is:



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Exercise 4.4.54

Exercise 4.4.54. Consider $y = f(x) = xe^{-x}$. Identity the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$. Indicate points where f' is 0 with horizontal hash marks and indicate points of inflection with X's.

Solution. First $f'(x) = [1](e^{-x}) + (x)[e^{-x}\overset{\curvearrowright}{[-1]}] = e^{-x}(1-x)$, so $x = 1$ is a critical point since $f'(1) = 0$. Next $f''(x) = [e^{-x}\overset{\curvearrowright}{[-1]}](1-x) + (e^{-x})[-1] = -e^{-x}((1-x)+1) = -e^{-x}(2-x)$ so $x = 2$ is a potential point of inflection. We perform a sign test on f' and f'':

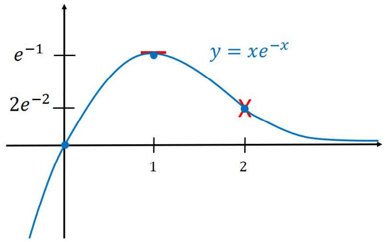
interval	$(-\infty,1)$	$(1,2)$	$(2,\infty)$
test value k	0	$3/2$	3
$f'(k)$	$e^{-(0)}(1-(0)) = 1$	$e^{-(3/2)}(1-(3/2)) = -(1/2)e^{-3/2}$	$e^{-(3)}(1-(3)) = -2e^{-3}$
$f'(x)$	$+$	$-$	$-$
$f''(k)$	$-e^{-(0)}(2-(0)) = -2$	$-e^{-(3/2)}(2-(3/2)) = -(1/2)e^{-3/2}$	$-e^{-(3)}(2-(3)) = e^{-3}$
$f''(x)$	$-$	$-$	$+$
$f(x)$	INC, CD	DEC, CD	DEC, CU

Exercise 4.4.54 (continued)

Solution (continued). . . .

interval	$(-\infty, 1)$	$(1, 2)$	$(2, \infty)$
$f(x)$	INC, CD	DEC, CD	DEC, CU

By the First Derivative Test for Local Extrema (Theorem 4.3.A), f has a local maximum at $x = 1$ of $f(1) = (1)e^{-(1)} = e^{-1}$. By definition, f has a point of inflection at $(2, f(2)) = (2, (2)e^{-(2)}) = (2, 2e^{-2})$. So the coordinates of the local maximum are $(1, e^{-1})$ and the coordinates of the point of inflection are $(2, 2e^{-2})$. Notice $f(0) = 0$ (notice that $xe^{-1} > 0$ for $x > 0$; we can show that $\lim_{x\to\infty} xe^{-x} = 0$ in the next section). The graph is:



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Exercise 4.4.74

Exercise 4.4.74. Let $y = f(x)$ be a continuous function with $y'(t) = \sin t$ for $t \in [0, 2\pi]$. Find y'' and then use Steps 2–4 of the graphing procedure to sketch the general shape of the graph of f. Indicate points where f' is 0 with horizontal hash marks.

Solution. First, if $y'(t) = \sin t$ then $y''(t) = \cos t$.

(2) We have y' and y'' above.

(3) Since $y'(t) = \sin t$ then the critical points of y for $t \in [0, 2\pi]$ are $t = 0$, $t = \pi$, and $t = 2\pi$, since y' is 0 at each of these.

(4) We perform a sign test on $y'(t) = \sin t$:

interval	$(0, \pi)$	$(\pi, 2\pi)$
test value k	$\pi/4$	$5\pi/4$
$f'(k)$	$\sin \pi/4 = \sqrt{2}/2$	$\sin 5\pi/4 = -\sqrt{2}/2$
$f'(x)$	$+$	$-$
$f(x)$	INC	DEC

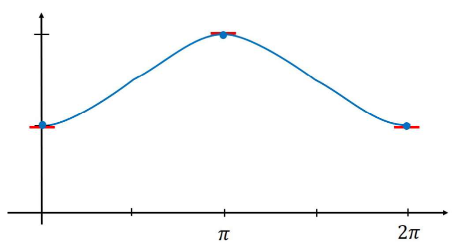
Exercise 4.4.74 (continued)

Exercise 4.4.74. Let $y = f(x)$ be a continuous function with $y'(t) = \sin t$ for $t \in [0, 2\pi]$. Find y'' and then use Steps 2–4 of the graphing procedure to sketch the general shape of the graph of f. Indicate points where f' is 0 with horizontal hash marks.

Solution (continued). . . .

interval	$(0, \pi)$	$(\pi, 2\pi)$
$f(x)$	INC	DEC

We don't know any function values (but suspect some periodic behavior since $y'(t) = \sin t$), and so have:



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Exercise 4.4.92

Exercise 4.4.92. Graph the rational function $y = f(x) = \dfrac{x^2 - 4}{x^2 - 2}$. Use all the steps in the graphing procedure. Indicate points where f' is 0 with horizontal hash marks and indicate points of inflection with X's.

Solution. We throw all of our graphing knowledge at this one!

(1) With $y = f(x) = \dfrac{x^2 - 4}{x^2 - 2}$, the domain is all $x \in \mathbb{R}$ except $x = \pm\sqrt{2}$ (since the denominator is 0 there). That is, the domain is $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)$. Notice that $f(-x) = \dfrac{(-x)^2 - 4}{(-x)^2 - 2} = \dfrac{x^2 - 4}{x^2 - 2} = f(x)$, so f is an even function and hence symmetric with respect to the y-axis.

Exercise 4.4.92 (continued 1)

Solution (continued). (2) We have

$y' = \dfrac{[2x](x^2-2)-(x^2-4)[2x]}{(x^2-2)^2} = \dfrac{4x}{(x^2-2)^2}$, and

$y'' = \dfrac{[4](x^2-2)^2-(4x)[2(x^2-2)[2x]]}{((x^2-2)^2)^2} = \dfrac{4(x^2-2)((x^2-2)-4x^2)}{(x^2-2)^4} = \dfrac{-4(3x^2+2)}{(x^2-2)^3}$.

(3) We see from $y' = \dfrac{4x}{(x^2-2)^2}$ that $\boxed{x=0 \text{ is the only critical point}}$ (since $\pm\sqrt{2}$ are not in the domain of f), and $f'(0)=0$. Notice $f(0) = \dfrac{(0)^2-4}{(0)^2-2} = 2$.

Exercise 4.4.92 (continued 2)

Solution (continued). (4) We perform a sign test on y' by removing the critical point from the domain of y:

interval	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},\infty)$
test value k	-2	-1	1	2
$f'(k)$	$\frac{4(-2)}{((-2)^2-2)^2}$	$\frac{4(-1)}{((-1)^2-2)^2}$	$\frac{4(1)}{((1)^2-2)^2}$	$\frac{4(2)}{((2)^2-2)^2}$
$f'(x)$	$-$	$-$	$+$	$+$
$f(x)$	DEC	DEC	INC	INC

So y is $\boxed{\text{decreasing on } (-\infty,-\sqrt{2})\cup(-\sqrt{2},0)}$ and y $\boxed{\text{increasing on } (0,\sqrt{2})\cup(\sqrt{2},\infty)}$.

(5) We see from $y'' = \dfrac{-4(3x^2+2)}{(x^2-2)^3}$ that y has no potential points of inflection (since the numerator is never 0 and the denominator is never 0 at points in the domain of y). So we perform a sign test on y'' on the domain of y...

Exercise 4.4.92 (continued 3)

Solution (continued).

interval	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},\sqrt{2})$	$(\sqrt{2},\infty)$
test value k	-2	0	2
$f''(k)$	$\frac{-4(3(-2)^2+2)}{((-2)^2-2)^3} = \frac{-56}{8}$	$\frac{-4(3(0)^2+2)}{((0)^2-2)^3} = \frac{-8}{-8}$	$\frac{-4(3(2)^2+2)}{((2)^2-2)^3} = \frac{-56}{8}$
$f''(x)$	$-$	$+$	$-$
$f(x)$	CD	CU	CD

So y is $\boxed{\text{CU on } (-\sqrt{2},\sqrt{2})}$ and y is $\boxed{\text{CD on } (-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)}$.

(6) Now for asymptotes. Notice

$$\lim_{x\to\pm\infty}\frac{x^2-4}{x^2-2} = \lim_{x\to\pm\infty}\frac{x^2-4}{x^2-2}\,\frac{1/x^2}{1/x^2} = \lim_{x\to\pm\infty}\frac{(x^2-4)/x^2}{(x^2-2)/x^2}$$

$$= \lim_{x\to\pm\infty}\frac{1-4/x^2}{1-2/x^2} = \frac{1-4\,(\lim_{x\to\pm\infty}1/x)^2}{1-2\,(\lim_{x\to\pm\infty}1/x)^2} = \frac{1-4(0)^2}{1-2(0)^2} = 1.$$

So $y=1$ is a horizontal asymptote.

Exercise 4.4.92 (continued 4)

Solution (continued). By Dr. Bob's Infinite Limits Theorem, $f(x) = \dfrac{x^2-4}{x^2-2} = \dfrac{x^2-4}{(x+\sqrt{2})(x-\sqrt{2})}$ satisfies $\lim_{x\to\pm\sqrt{2}} f(x) = \pm\infty$ and so f has $\boxed{\text{vertical asymptotes at } x=\pm\sqrt{2}}$. We consider the four sign diagrams: (1) For $x\to-\sqrt{2}^-$ (so that x is "close to" $-\sqrt{2}$ and less than $-\sqrt{2}$) we have $\dfrac{x^2-4}{(x+\sqrt{2})(x-\sqrt{2})} \Longrightarrow \dfrac{(-)}{(-)(-)} = -$, (2) for $x\to-\sqrt{2}^+$ we have $\dfrac{x^2-4}{(x+\sqrt{2})(x-\sqrt{2})} \Longrightarrow \dfrac{(-)}{(+)(-)} = +$, (3) for $x\to\sqrt{2}^-$ we have $\dfrac{x^2-4}{(x+\sqrt{2})(x-\sqrt{2})} \Longrightarrow \dfrac{(-)}{(+)(-)} = +$, and (4) for $x\to\sqrt{2}^+$ we have $\dfrac{x^2-4}{(x+\sqrt{2})(x-\sqrt{2})} \Longrightarrow \dfrac{(-)}{(+)(+)} = -$. So $\lim_{x\to-\sqrt{2}^-} f(x) = -\infty$, $\lim_{x\to-\sqrt{2}^+} f(x) = \infty$, $\lim_{x\to\sqrt{2}^-} f(x) = \infty$, and $\lim_{x\to\sqrt{2}^+} f(x) = -\infty$.

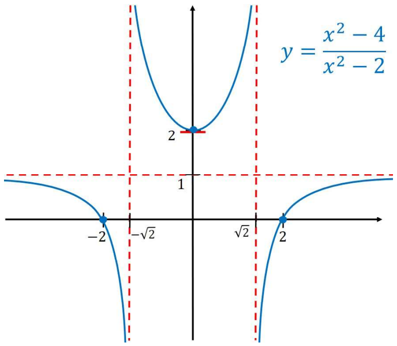
Exercise 4.4.92 (continued 5)

Solution (continued).

(7) We have:

interval	$(-\infty,-\sqrt{2})$	$(-\sqrt{2},0)$	$(0,\sqrt{2})$	$(\sqrt{2},\infty)$
$f(x)$	DEC, CD	DEC, CU	INC, CU	INC, CD

Since $y = f(x) = \dfrac{x^2-4}{x^2-2}$, then the y-intercepts are $x = \pm 2$. We have $f(0) = 2$ from above. So...



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Exercise 4.4.122

Exercise 4.4.122. Parabolas.
(a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, where $a \neq 0$. **(b)** When is the parabola concave up? Concave down? Give reasons for your answer.

Solution. We have $y' = 2ax + b$ and $y'' = 2a$.

(a) The vertex of a parabola $y = ax^2 + bx + c$ is an absolute extreme of the function $f(x) = ax^2 + bx + c$, and hence a local extreme value. So by Theorem 4.2, "Local Extreme Values," the vertex occurs at a critical point of f. Since $f'(x) = 2ax + b$ then the only critical point is $x = -b/(2a)$.
Since $f\left(\dfrac{-b}{2a}\right) = a\left(\dfrac{-b}{2a}\right)^2 + b\left(\dfrac{-b}{2a}\right) + c = \dfrac{b^2}{4a} - \dfrac{b^2}{2a} + c = \dfrac{b^2}{4a} - \dfrac{2b^2}{4a} + \dfrac{4ac}{4a} = \dfrac{-b^2+4ac}{4a}$ So the coordinates of the vertex is $\boxed{(-b/(2a), f(-b/(2a))) = (-b/a, (-b^2+4ac)/(4a))}$.

Exercise 4.4.122 (continued)

Exercise 4.4.122. Parabolas.
(a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, where $a \neq 0$. **(b)** When is the parabola concave up? Concave down? Give reasons for your answer.

Solution (continued). (b) Since $y'' = f''(x) = 2a$ then by the Second Derivative Test for Concavity (Theorem 4.4.A), the parabola is $\boxed{\text{concave up everywhere when } a > 0}$ and the parabola is $\boxed{\text{concave down everywhere when } a < 0}$. □

Note. Notice that the second degree polynomial function $f(x) = ax^2 + bx + c$ has no point of inflection.

Exercise 4.4.124

Exercise 4.4.124. Cubic Curves.
What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$? Give reasons for your answer.

Solution. We have $y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$. With $y = f(x)$ we have $f''(-b/(3a)) = 0$ then $-b/(3a)$ is a potential point of inflection. So we perform a sign test on $f''(x)$:

interval	$(-\infty,-b/(3a))$	$(-b/(3a),\infty)$
test value k	$-b/(3a)-1$	$-b/(3a)+1$
$f''(k)$	$6a(-b/(3a)-1)+2b = -6a$	$6a(-b/(3a)+1)+2b = 6a$

Since $a \neq 0$ by hypothesis, then by the Second Derivative Test for Concavity (Theorem 4.4.A) f changes concavity at $x = -b/(3a)$. So there is only one inflection point and, since...

Exercise 4.4.124 (continued)

Exercise 4.4.124. Cubic Curves.
What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$? Give reasons for your answer.

Solution (continued). So there is only one inflection point and, since

$$f\left(\frac{-b}{3a}\right) = a\left(\frac{-b}{3a}\right)^3 + b\left(\frac{-b}{3a}\right)^2 + c\left(\frac{-b}{3a}\right) + c$$

$$= \frac{-b^3}{27a^2} + \frac{b^3}{9a^2} + \frac{-bc}{3a} + c = \frac{-b^3}{27a^2} + \frac{3b^3}{27a^2} + \frac{-9abc}{27a^2} + \frac{27a^2c}{27a^2}$$

$$= \frac{2b^3 - 9abc + 27a^2c}{27a^2},$$

the $\boxed{\text{inflection point is } \left(\frac{-b}{3a}, \frac{2b^3 - 9abc + 27a^2c}{27a^2}\right)}$. □