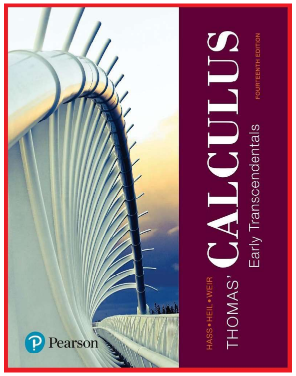
Calculus 1

Chapter 4. Applications of Derivatives

4.5. Indeterminate Forms and L'Hôpital's Rule—Examples and Proofs



Exercise 4.5.A

Example 4.5.A. Use l'Hôpital's Rule to show **(a)** $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and **(b)** $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. (a) With $f(x) = \sin x$ and $g(x) = x$ and $a = 0$, we have $f(a) = g(a) = 0$, $f'(x) = \cos x$, and $g'(x) = 1$. So the hypotheses of l'Hôpital's Rule hold and hence we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin x}{x} \stackrel{0/0}{=} \frac{f'(a)}{g'(a)} = \frac{\cos(0)}{(1)} = \frac{1}{1} = \boxed{1}. \quad \square$$

(b) With $f(x) = 1-\cos x$ and $g(x) = x$ and $a = 0$, we have $f(a) = g(a) = 0$, $f'(x) = \sin x$, and $g'(x) = 1$. So the hypotheses of l'Hôpital's Rule hold and hence we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{1-\cos x}{x} \stackrel{0/0}{=} \frac{f'(a)}{g'(a)} = \frac{\sin(0)}{(1)} = \frac{0}{1} = \boxed{0}. \quad \square$$

Exercise 4.5.16

Exercise 4.5.16. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0}\frac{\sin x - x}{x^3}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With $f(x) = \sin x - x$ and $g(x) = x^3$ and $a = 0$, we have $f(a) = g(a) = 0$, $f'(x) = \cos x - 1$, and $g'(x) = 3x^2$. So the hypotheses of l'Hôpital's Rule hold, but $f'(a)/g'(a)$ does not exist since $g'(0) = 0$. However, $\lim_{x\to 0} f'(x)/g'(x)$ is itself of the 0/0 indeterminate form so that we may attempt to apply l'Hôpital's Rule to that (or maybe even $\lim_{x\to 0} f''(x)/g''(x)$). We then have

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} \stackrel{0/0}{=} \lim_{x\to 0}\frac{\cos x - 1}{3x^2} \stackrel{0/0}{=} \lim_{x\to 0}\frac{-\sin x}{6x}$$

$$\stackrel{0/0}{=} \lim_{x\to 0}\frac{-\cos x}{6} = \frac{-\cos(0)}{6} = \boxed{\frac{-1}{6}}. \quad \square$$

Exercise 4.5.38

Exercise 4.5.38. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0^+}(\ln x - \ln \sin x)$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. First, we rewrite the function $\ln x - \ln\sin x$ as $\ln\frac{x}{\sin x}$. Then

$$\begin{aligned}
\lim_{x\to 0^+}(\ln x - \ln\sin x) &= \lim_{x\to 0^+}\ln\frac{x}{\sin x} \\
&= \ln\left(\lim_{x\to 0^+}\frac{x}{\sin x}\right) \text{ since } \ln x \text{ is continuous} \\
&\stackrel{0/0}{=} \ln\left(\lim_{x\to 0^+}\frac{1}{\cos x}\right) \\
&= \ln\left(\frac{1}{\cos(0)}\right) \text{ since } \cos x \text{ is continuous at } 0 \\
&= \ln(1) = \boxed{0}. \quad \square
\end{aligned}$$

Exercise 4.5.46

Exercise 4.5.46. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to\infty} x^2e^{-x}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Proof. With $f(x) = e^{-x}$ and $g(x) = x^2$ we have $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{-x} = \lim_{x\to-\infty} e^x = 0$ by Example 2.6.5 (where we have replaced x with $-x$). So $\lim_{x\to\infty} x^2e^{-x}$ is of the $0\cdot\infty$ indeterminate form. We rewrite the function x^2e^{-x} as x^2/e^x and note that $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to\infty} e^x = \infty$, so that $\lim_{x\to\infty} x^2/e^x$ is of the ∞/∞ indeterminate form. So we have by Theorem 4.5.A, "L'Hôpitals Rule for ∞/∞ Indeterminate Forms," that

$$\begin{aligned}\lim_{x\to\infty} x^2e^{-x} &= \lim_{x\to\infty}\frac{x^2}{e^x} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{2x}{e^x} \overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{2}{e^x}\\ &= 2\lim_{x\to\infty} e^{-x} = 2(0) = \boxed{0} \text{ by Example 2.6.5.} \quad\square\end{aligned}$$

Exercise 4.5.32

Exercise 4.5.32. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With $f(x) = \log_2 x$ and $g(x) = \log_3(x+3)$, we have $\lim_{x\to\infty}\log_2 x = \lim_{x\to\infty}\log_3(x+3) = \infty$, so $\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}$ is of the ∞/∞ indeterminate form. So by Theorem 4.5.A, "L'Hôpitals Rule for ∞/∞ Indeterminate Forms,"

$$\begin{aligned}\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)} &\overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{(1/\ln 2)(1/x)}{(1/\ln 3)(1/(x+3))}\\ &= \frac{\ln 3}{\ln 2}\lim_{x\to\infty}\frac{x+3}{x} \overset{\infty/\infty}{=} \frac{\ln 3}{\ln 2}\lim_{x\to\infty}\frac{1}{1} = \boxed{\frac{\ln 3}{\ln 2}}. \quad\square\end{aligned}$$

Exercise 4.5.40

Exercise 4.5.40. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With $f(x) = (3x+1)/x$ and $g(x) = 1/\sin x = \csc x$ we have $\lim_{x\to 0^+}(3x+1)/x = \infty$ (by Dr. Bob's Infinite Limits Theorem) and $\lim_{x\to 0^+}\csc x = \infty$ (see the graph of $y = \csc x$), so $\lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$ is of the $\infty - \infty$ indeterminate form. So we get a common denominator as follows

$$\lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+}\left(\frac{(3x+1)\sin x}{x\sin x} - \frac{x}{x\sin x}\right)$$

$$= \lim_{x\to 0^+}\frac{(3x+1)\sin x - x}{x\sin x} \overset{0/0}{=} \lim_{x\to 0^+}\frac{[3](\sin x) + (3x+1)[\cos x] - 1}{[1](\sin x) + (x)[\cos x]}$$

Exercise 4.5.40 (continued)

Exercise 4.5.40. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0^+}\left(\frac{3x+1}{x} - \frac{1}{\sin x}\right)$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution (continued). ...

$$= \lim_{x\to 0^+}\frac{[3](\sin x) + (3x+1)[\cos x] - 1}{[1](\sin x) + (x)[\cos x]} = \lim_{x\to 0^+}\frac{3\sin x + (3x+1)\cos x - 1}{\sin x + x\cos x}$$

$$\overset{0/0}{=} \lim_{x\to 0^+}\frac{3\cos x + [3](\cos x) + (3x+1)[-\sin x]}{\cos x + [1](\cos x) + (x)[-\sin x]}$$

$$= \lim_{x\to 0^+}\frac{6\cos x - (3x+1)\sin x}{2\cos x - x\sin x} = \frac{6\cos(0) - (3(0)+1)\sin(0)}{2\cos(0) - (0)\sin(0)} = \frac{6}{2} = \boxed{3}. \square$$

Theorem 4.5.B

Theorem 4.5.B. If $\lim_{x\to a} \ln f(x) = L$ then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a} \ln f(x)} = e^L.$$

Here, a may be finite or infinite.

Proof. Suppose $\lim_{x\to a} \ln f(x) = L$. Then by the definition of limit, $\ln f(x)$ is defined on some open interval I containing a, except possibly at a itself. Since the natural logarithm is the inverse of the natural exponential, then $e^{\ln f(x)} = f(x)$ on I except possibly at $x = a$. Since the natural exponential function is continuous everywhere (in particular, at L) then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^{\lim_{x\to a} \ln f(x)} = e^L,$$

as claimed. □

Exercise 4.5.52

Exercise 4.5.52. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 1^+} x^{1/(x-1)}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With $f(x) = x$ and $g(x) = 1/(x-1)$, we have $\lim_{x\to 1^+} f(x) = 1$ and $\lim_{x\to 1^+} g(x) = \infty$, so $\lim_{x\to 1^+} x^{1/(x-1)}$ is of the 1^∞ indeterminate form. We take a natural logarithm to get

$$\lim_{x\to 1^+} \ln(x^{1/(x-1)}) = \lim_{x\to 1^+} \frac{1}{x-1} \ln x = \lim_{x\to 1^+} \frac{\ln x}{x-1}$$

$$\overset{0/0}{=} \lim_{x\to 1^+} \frac{1/x}{1} = \frac{1/(1)}{1} = 1.$$

So by Theorem 4.5.B, $\lim_{x\to 1^+} x^{1/(x-1)} = e^{\lim_{x\to 1^+} \ln x^{1/(x-1)}} = e^1 = \boxed{e}$. □

Exercise 4.5.58

Exercise 4.5.58. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0} (e^x + x)^{1/x}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With $f(x) = e^x + x$ and $g(x) = 1/x$, we have $\lim_{x\to 0} f(x) = 1$, $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} 1/x = \infty$, and $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} 1/x = -\infty$, so both $\lim_{x\to 0^+}(e^x + x)^{1/x}$ and $\lim_{x\to 0^-}(e^x + x)^{1/x}$ are of the 1^∞ indeterminate form. To evaluate $\lim_{x\to 0^+}(e^x + x)^{1/x}$, we take a natural logarithm to get

$$\lim_{x\to 0^+} \ln(e^x + x)^{1/x} = \lim_{x\to 0^+} (1/x)\ln(e^x + x) = \lim_{x\to 0^+} \frac{\ln(e^x + x)}{x}$$

$$\overset{0/0}{=} \lim_{x\to 0^+} \frac{(1/(e^x + x))\overset{\frown}{}[e^x + 1]}{[1]} = \lim_{x\to 0^+} \frac{e^x + 1}{e^x + x} = \frac{e^{(0)} + 1}{e^{(0)} + (0)} = 2.$$

Exercise 4.5.58 (continued)

Exercise 4.5.58. Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to 0} (e^x + x)^{1/x}$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution (continued). So by Theorem 4.5.B, $\lim_{x\to 0^+} (e^x + x)^{1/x} = e^{\lim_{x\to 0^+} \ln(e^x+x)^{1/x}} = e^2$. To evaluate $\lim_{x\to 0^-}(e^x + x)^{1/x}$, we take a natural logarithm to get

$$\lim_{x\to 0^-} \ln(e^x + x)^{1/x} = \lim_{x\to 0^-} (1/x)\ln(e^x + x) = \lim_{x\to 0^-} \frac{\ln(e^x + x)}{x}$$

$$\overset{0/0}{=} \lim_{x\to 0^-} \frac{(1/(e^x + x))\overset{\frown}{}[e^x + 1]}{[1]} = \lim_{x\to 0^-} \frac{e^x + 1}{e^x + x} = \frac{e^{(0)} + 1}{e^{(0)} + (0)} = 2.$$

So by Theorem 4.5.B, $\lim_{x\to 0^-} (e^x + x)^{1/x} = e^{\lim_{x\to 0^-} \ln(e^x+x)^{1/x}} = e^2$. Therefore by Theorem 2.6, "Relation Between One-Sided and Two-Sided Limits," $\lim_{x\to 0}(e^x + x)^{1/x} = \boxed{e^2}$. □

Exercise 4.5.81(b)

Exercise 4.5.81(b). Use l'Hôpital's Rule (Theorem 4.6) to evaluate $\lim_{x\to\infty}(x-\sqrt{x^2+x})$. Write the indeterminate form over the equal sign when you use l'Hôpital's Rule. HINT: As the first step, multiply by $(x+\sqrt{x^2+x})/(x+\sqrt{x^2+x})$ and simplify the new numerator.

Solution. Notice that $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty}\sqrt{x^2+x} = \infty$, so that $\lim_{x\to\infty}(x-\sqrt{x^2+x})$ is of an $\infty-\infty$ indeterminate form. We follow the hint and consider

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\left(\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right)$$

$$= \lim_{x\to\infty}\frac{(x)^2-(\sqrt{x^2+x})^2}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$$

Exercise 4.5.81(b) (continued)

Solution (continued).

$$\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}}$$

$$\overset{\infty/\infty}{=} \lim_{x\to\infty}\frac{-1}{1+(1/2)(x^2+x)^{-1/2}[2x+1]}$$

$$= \lim_{x\to\infty}\frac{-1}{1+(2x+1)/(2\sqrt{x^2+x})} = \frac{-1}{1+(1)} = \boxed{\frac{-1}{2}}$$

because

$$\lim_{x\to\infty}\frac{2x+1}{2\sqrt{x^2+x}} = \lim_{x\to\infty}\frac{2x+1}{2\sqrt{x^2+x}}\,\frac{1/x}{1/x} = \lim_{x\to\infty}\frac{(2x+1)/x}{2\sqrt{x^2+x}/\sqrt{x^2}}$$

$$= \lim_{x\to\infty}\frac{2+1/x}{2\sqrt{1+1/x}} = \frac{2+(0)}{2\sqrt{1+(0)}} = 1. \qquad \square$$

Theorem 4.7

Theorem 4.7. Cauchy's Mean Value Theorem.
Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $g'(x)\neq 0$ throughout (a,b). Then there exists a number c in (a,b) at which $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. First, notice that f and g both satisfy the hypotheses of the Mean Value Theorem (Theorem 4.4). We claim that $g(a)\neq g(b)$, for if $g(a)=g(b)$ then by the the Mean Value Theorem we have $g'(c) = \dfrac{g(b)-g(a)}{b-a} = 0$ for some $c\in(a,b)$ contradicting the hypotheses of the theorem. Next, consider

$$F(x) = f(x)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)).$$

Since f and g are continuous on $[a,b]$ then so is F, since f and g are differentiable on (a,b) then so is F, and $F(a)=F(b)=0$.

Theorem 4.7 (continued)

Theorem 4.7. Cauchy's Mean Value Theorem.
Suppose functions f and g are continuous on $[a,b]$ and differentiable throughout (a,b) and also suppose $g'(x)\neq 0$ throughout (a,b). Then there exists a number c in (a,b) at which $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$.

Proof (continued). Since f and g are continuous on $[a,b]$ then so is F, since f and g are differentiable on (a,b) then so is F, and $F(a)=F(b)=0$. So by Rolle's Theorem (Theorem 4.3) there is $c\in(a,b)$ such that $F'(c)=0$. Since $F'(x) = f'(x) - \dfrac{f(b)-f(a)}{g(b)-g(a)}g'(x)$, then $F'(c) = f'(c) - \dfrac{f(b)-f(a)}{g(b)-g(a)}g'(c) = 0$ and hence $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$, as claimed. $\square$

Theorem 4.6

Theorem 4.6. L'Hôpital's Rule.
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Proof. We consider one-sided limits. Suppose $x \to a^+$ and $x \in I$. Then $g'(x) \neq 0$, so by Cauchy's Mean Value Theorem (Theorem 4.7) applied on the interval $[a, x]$ we have for some $c \in (a, x)$ that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(x) - f(a)}{g(x) - g(a)}$. Since $f(a) = g(a) = 0$ by hypothesis, then $\dfrac{f'(c)}{g'(c)} = \dfrac{f(x)}{g(x)}$. Notice that as $x \to a^+$ then $c \to a^+$ (since for any given x, the corresponding c is between a and x).

Theorem 4.6 (continued)

Theorem 4.6. L'Hôpital's Rule.
Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

Proof (continued). Therefore

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

so l'Hôpital's Rule holds as $x \to a^+$. The same argument (except with Cauchy's Mean Value Theorem applied on the interval $[x, a]$) shows that l'Hôpital's Rule holds as $x \to a^-$ also. So by Theorem 2.6, "Relation Between One-Sided and Two-Sided Limits," the claim holds. □