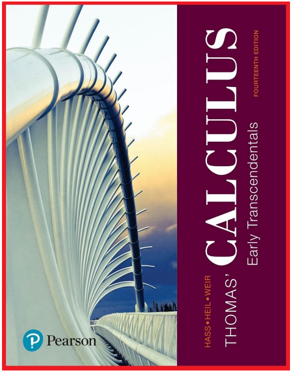
Calculus 1

Chapter 4. Applications of Derivatives

4.6. Applied Optimization—Examples and Proofs



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Exercise 4.6.2

Exercise 4.6.2. Show that among all rectangles with an 8-m perimeter, the one with the largest area is a square.

Solution. First, we consider a rectangle of width x and height y:

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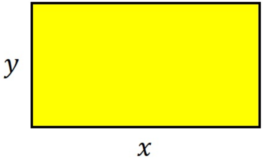

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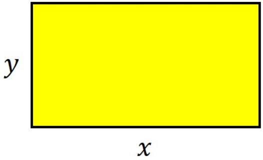

In terms of x and y, the perimeter is $2x + 2y$ so that we must have $2x + 2y = 8$ m or $x + y = 4$ m or $y = 4 - x$. Since x and y are distances (and hence non-negative), then we must have $x \in [0, 4]$.

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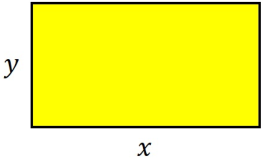

In terms of x and y, the perimeter is $2x + 2y$ so that we must have $2x + 2y = 8$ m or $x + y = 4$ m or $y = 4 - x$. Since x and y are distances (and hence non-negative), then we must have $x \in [0, 4]$. In terms of x and y, the area of the rectangle is $A = xy$. So as a function of x, we have $A(x) = xy = x(4 - x)$.

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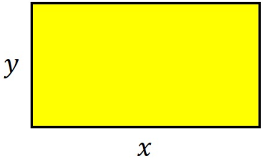

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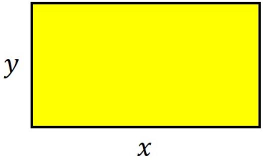

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Exercise 4.6.2 (continued)

Exercise 4.6.2. Show that among all rectangles with an 8-m perimeter, the one with the largest area is a square.

Solution. What is the maximum of $A(x) = x(4 - x) = 4x - x^2$ for $x \in [0, 4]$? We have $A'(x) = 4 - 2x$, so the critical point is $x = 2$. As in Section 4.1 (using the technique based on Theorem 4.2, "Local Extreme Values"), we test the endpoints and critical points: $A(0) = 4(0) - (0)^2 = 0$, $A(2) = 4(2) - (2)^2 = 4$, and $A(4) = 4(4) - (4)^2 = 0$. So the maximum area is 4 m^2 and occurs when $x = 2$ m and $y = 4 - x = 4 - (2) = 2$ m.

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Exercise 4.6.8

Exercise 4.6.8. The Shortest Fence.
A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Solution. (2 and 3) We consider a rectangle with width x and height y. We divide the field in half as follows:

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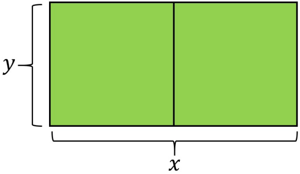

Since the area of the field is 216 m^2, then we have $xy = 216$ m^2, or $y = 216/x$.

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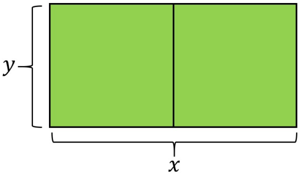

Since the area of the field is 216 m^2, then we have $xy = 216$ m^2, or $y = 216/x$.

Exercise 4.6.8 (continued 1)

Solution (continued). (4) The amount of fencing in terms of x and y is $F = 2x + 3y$. So as a function of x we have
$F(x) = 2x + 3(216/x) = 2x + 648/x = 2x + 648x^{-1}$.

(5) The question is: What is the minimum of F for $x \in (0, \infty)$. We have $F'(x) = 2 - 648/x^2 = (2x^2 - 648)/x^2 = 2(x^2 - 324)/x^2$. So the critical points are $x = \pm\sqrt{324} = \pm 18$, but only the critical point $x = 18$ is in the interval $(0, \infty)$.

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interval	$(0, 18)$	$(18, \infty)$
test value k	1	20
$F'(k)$	$2((1)^2 - 324)/(1)^2 = -646$	$2((20)^2 - 324)/(20)^2 = 76/400$
$F'(x)$	$-$	$+$
$F(x)$	DEC	INC

So F has a local minimum at $x = 18$.

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Solution (continued). So F has a local minimum at $x = 18$. Since F has only one critical point in $(0, \infty)$ then the minimum at $x = 18$ must be an absolute minimum of F. When $x = 18$ then $y = 216/(18) = 12$ m and $F(18) = 2(18) + 648/(18) = 72$ m. That is, the dimensions of the field are 12 m by 18 m and the smallest amount of fence is 72 m. □

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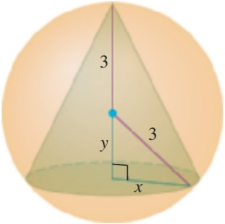
Exercise 4.6.12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



Solution. We follow the steps.

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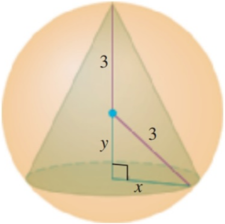

Solution. We follow the steps.

(2 and 3) We use the picture in the book. With the book's variables, the height of the cone is $h = y + 3$ and the radius is $r = x$. From the Pythagorean Theorem, $x^2 + y^2 = 3^2$ or $x^2 = 9 - y^2$. Since x and y are distances (and hence non-negative), then we must have $y \in [0, 3]$ (one might argue that $y \in [-3, 3]$, but a maximum volume clearly does not happen for $y \in [-3, 0)$).

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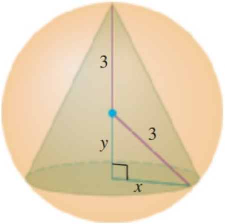

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(4) The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, so $V = \frac{1}{3}\pi x^2(y + 3)$, or $V(y) = \frac{1}{3}\pi(9 - y^2)(y + 3) = \frac{1}{3}\pi(-y^3 - 3y^2 + 9y + 27)$.

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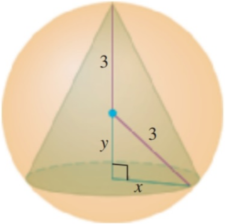

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Exercise 4.6.12 (continued)

Solution (continued). (5) The question is: Find the maximum of $V(y) = \frac{1}{3}\pi(-y^3 - 3y^2 + 9y + 27)$ for $y \in [0, 3]$. We have $V'(y) = \frac{1}{3}\pi(-3y^2 - 6y + 9) = \pi(-y^2 - 2y + 3) = \pi(-y + 1)(y + 3)$, and the critical points of V are $y = 1$ and $y = -3$. So as in Section 4.1, we consider the endpoints and critical points in $[0, 3]$. We have
$V(0) = \frac{1}{3}\pi(-(0)^3 - 3(0)^2 + 9(0) + 27) = 9\pi$,
$V(1) = \frac{1}{3}\pi(-(1)^3 - 3(1)^2 + 9(1) + 27) = \frac{32}{3}\pi$, and
$V(3) = \frac{1}{3}\pi(-(3)^3 - 3(3)^2 + 9(3) + 27) = 0$. So the maximum volume is $32\pi/3$ (cubic units). Notice that the occurs when the $y = 1$ and $x = \sqrt{9 - y^2} = \sqrt{9 - (1)^2} = \sqrt{8} = 2\sqrt{2}$. □

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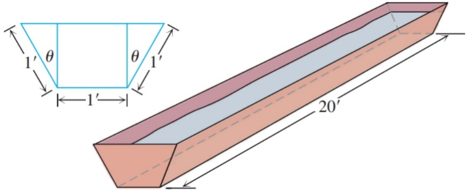
Exercise 4.6.24

Exercise 4.6.24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



Solution. (2 and 3) We start with the picture in the book. With the book's variable, we have:

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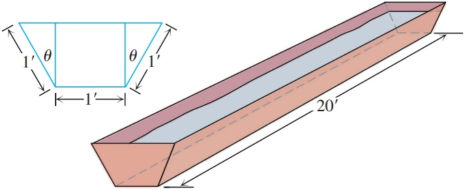

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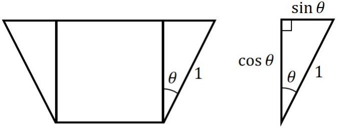

Since θ is in a right triangle then $\theta \in [0, \pi/2]$.

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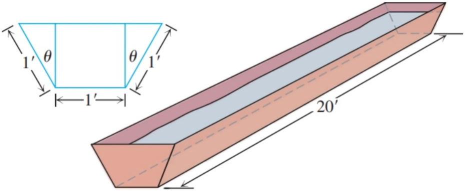

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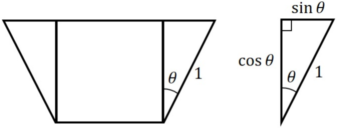

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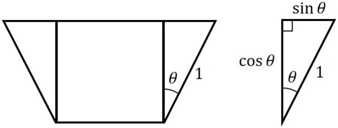
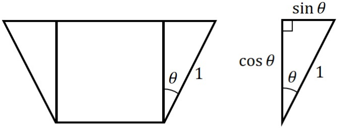


Solution. (4) We need the volume of the trough. We can think of the cross-section of the trough as two triangles, each of base $\sin\theta$ and height $\cos\theta$, and a rectangle of base 1 and height $\cos\theta$. So the cross-sectional area of the trough is $2\left(\frac{1}{2}\sin\theta\cos\theta\right)+(1)(\cos\theta)=\sin\theta\cos\theta+\cos\theta$ ft^2. Hence the volume of the trough is $V(\theta)=20(\sin\theta\cos\theta+\cos\theta)$ ft^3.

(5) The question is: Find the maximum of $V(\theta)=20(\sin\theta\cos\theta+\cos\theta)$ for $\theta\in[0,\pi/2]$. We have
$V'(\theta)=20([\cos\theta](\cos\theta)+(\sin\theta)[-\sin\theta]+[-\sin\theta])=$
$20(\cos^2\theta-\sin^2\theta-\sin\theta)=20((1-\sin^2\theta)-\sin^2\theta-\sin\theta)=$
$20(1-\sin\theta-2\sin^2\theta)=20(1+\sin\theta)(1-2\sin\theta)$.

Exercise 4.6.24 (continued 1)



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$V'(\theta)=20([\cos\theta](\cos\theta)+(\sin\theta)[-\sin\theta]+[-\sin\theta])=$
$20(\cos^2\theta-\sin^2\theta-\sin\theta)=20((1-\sin^2\theta)-\sin^2\theta-\sin\theta)=$
$20(1-\sin\theta-2\sin^2\theta)=20(1+\sin\theta)(1-2\sin\theta)$.

Exercise 4.6.24 (continued 2)

Solution. (5) We have $V'(\theta) = 20(1 + \sin\theta)(1 - 2\sin\theta)$. So the relevant critical points of V are $\theta \in [0, \pi/2]$ for which $\sin\theta = -1$ or $\sin\theta = 1/2$. There are no $\theta \in [0, \pi/2]$ such that $\sin\theta = -1$. For $\sin\theta = 1/2$ and $\theta \in [0, \pi/2]$ we have $\theta = \pi/6$, so that the only critical point in $[0, \pi/2]$ is $\pi/6$. So as in Section 4.1, we consider the endpoints and critical points in $[0, \pi/2]$.

Exercise 4.6.24 (continued 2)

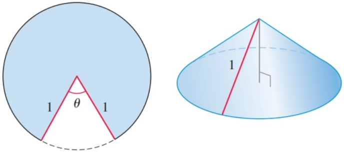
Solution. (5) We have $V'(\theta) = 20(1 + \sin\theta)(1 - 2\sin\theta)$. So the relevant critical points of V are $\theta \in [0, \pi/2]$ for which $\sin\theta = -1$ or $\sin\theta = 1/2$. There are no $\theta \in [0, \pi/2]$ such that $\sin\theta = -1$. For $\sin\theta = 1/2$ and $\theta \in [0, \pi/2]$ we have $\theta = \pi/6$, so that the only critical point in $[0, \pi/2]$ is $\pi/6$. So as in Section 4.1, we consider the endpoints and critical points in $[0, \pi/2]$. We have $V(0) = 20(\sin(0)\cos(0) + \cos(0)) = 20$, $V(\pi/6) = 20(\sin(\pi/6)\cos(\pi/6) + \cos(\pi/6)) = 20((1/2)(\sqrt{3}/2) + \sqrt{3}/2) = 20(\sqrt{3}/4 + \sqrt{3}/2) = 5\sqrt{3} + 10\sqrt{3} \approx 25.98$, and $V(\pi/2) = 20(\sin(\pi/2)\cos(\pi/2) + \cos(\pi/2)) = 0$. So the maximum of V occurs when $\theta = \pi/6$ and the maximum volume is $5\sqrt{3} + 10\sqrt{3}$ ft^3. $\square$

Exercise 4.6.24 (continued 2)

Solution. (5) We have $V'(\theta) = 20(1 + \sin\theta)(1 - 2\sin\theta)$. So the relevant critical points of V are $\theta \in [0, \pi/2]$ for which $\sin\theta = -1$ or $\sin\theta = 1/2$. There are no $\theta \in [0, \pi/2]$ such that $\sin\theta = -1$. For $\sin\theta = 1/2$ and $\theta \in [0, \pi/2]$ we have $\theta = \pi/6$, so that the only critical point in $[0, \pi/2]$ is $\pi/6$. So as in Section 4.1, we consider the endpoints and critical points in $[0, \pi/2]$. We have $V(0) = 20(\sin(0)\cos(0) + \cos(0)) = 20$, $V(\pi/6) = 20(\sin(\pi/6)\cos(\pi/6) + \cos(\pi/6)) = 20((1/2)(\sqrt{3}/2) + \sqrt{3}/2) = 20(\sqrt{3}/4 + \sqrt{3}/2) = 5\sqrt{3} + 10\sqrt{3} \approx 25.98$, and $V(\pi/2) = 20(\sin(\pi/2)\cos(\pi/2) + \cos(\pi/2)) = 0$. So the $\boxed{\text{maximum of } V \text{ occurs when } \theta = \pi/6}$ and the maximum volume is $5\sqrt{3} + 10\sqrt{3}$ ft^3. $\square$

Exercise 4.6.42

Exercise 4.6.42. A cone is formed from a circular piece of material of radius 1 meter by removing a section of angle θ and then joining the two straight edges. Determine the largest possible volume for the cone.



Solution. (2 and 3) We use the picture in the book, but we introduce height h and radius r of the cone as variables. Notice that $h^2 + r^2 = 1^2$ or $h = \sqrt{1 - r^2}$.

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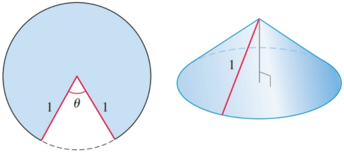

Solution. (2 and 3) We use the picture in the book, but we introduce height h and radius r of the cone as variables. Notice that $h^2 + r^2 = 1^2$ or $h = \sqrt{1 - r^2}$. The circumference of the base of the cone is $2\pi r$, and this equals the length of the arc subtended by the angle $2\pi - \theta$ in the circle of radius 1 (namely, the length $(2\pi - \theta)(1) = 2\pi - \theta$); notice that $\theta \in [0, 2\pi]$. So we have $2\pi r = 2\pi - \theta$ or $r = 1 - \theta/(2\pi)$.

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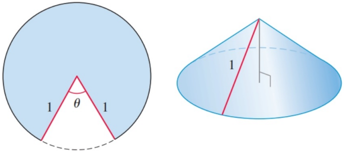

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Exercise 4.6.42 (continued 1)

Solution (continued). Notice that this gives $h = \sqrt{1-r^2} = \sqrt{1-(1-\theta/(2\pi))^2} = \sqrt{1-(1-\theta/\pi+\theta^2/(4\pi^2))} = \sqrt{\theta/\pi-\theta^2/(4\pi^2)}$.

(4) The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. In terms of θ,

$$V(\theta) = \frac{1}{3}\pi(1-\theta/(2\pi))^2\sqrt{\theta/\pi-\theta^2/(4\pi^2)} = \frac{\pi}{3}\left(\frac{2\pi-\theta}{2\pi}\right)^2\sqrt{\frac{4\pi\theta-\theta^2}{4\pi^2}}$$

$$= \frac{\pi}{24\pi^3}(2\pi-\theta)^2\sqrt{4\pi\theta-\theta^2} = \frac{1}{24\pi^2}(2\pi-\theta)^2(4\pi\theta-\theta^2)^{1/2}.$$

Exercise 4.6.42 (continued 1)

Solution (continued). Notice that this gives $h = \sqrt{1 - r^2} = \sqrt{1 - (1 - \theta/(2\pi))^2} = \sqrt{1 - (1 - \theta/\pi + \theta^2/(4\pi^2))} = \sqrt{\theta/\pi - \theta^2/(4\pi^2)}$.

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(5) The question is: Find the maximum of $V(\theta) = \frac{1}{24\pi^2}(2\pi - \theta)^2(4\pi\theta - \theta^2)^{1/2}$ for $\theta \in [0, 2\pi]$.

Exercise 4.6.42 (continued 1)

Solution (continued). Notice that this gives $h = \sqrt{1 - r^2} = \sqrt{1 - (1 - \theta/(2\pi))^2} = \sqrt{1 - (1 - \theta/\pi + \theta^2/(4\pi^2))} = \sqrt{\theta/\pi - \theta^2/(4\pi^2)}$.

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(5) The question is: Find the maximum of $V(\theta) = \frac{1}{24\pi^2}(2\pi - \theta)^2(4\pi\theta - \theta^2)^{1/2}$ for $\theta \in [0, 2\pi]$. We have

$$V'(\theta) = \frac{1}{24\pi^2}\left([2(2\pi - \overset{\curvearrowright}{\theta})[-1]](4\pi\theta - \theta^2)^{1/2}\right.$$

$$\left. +(2\pi - \theta)^2\left[\frac{1}{2}(4\pi\theta - \theta^2)^{-1/\overset{\curvearrowright}{2}}[4\pi - 2\theta]\right]\right)$$

Exercise 4.6.42 (continued 1)

Solution (continued). Notice that this gives $h = \sqrt{1 - r^2} = \sqrt{1 - (1 - \theta/(2\pi))^2} = \sqrt{1 - (1 - \theta/\pi + \theta^2/(4\pi^2))} = \sqrt{\theta/\pi - \theta^2/(4\pi^2)}$.

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(5) The question is: Find the maximum of $V(\theta) = \frac{1}{24\pi^2}(2\pi - \theta)^2(4\pi\theta - \theta^2)^{1/2}$ for $\theta \in [0, 2\pi]$. We have

$$V'(\theta) = \frac{1}{24\pi^2}\Bigg([2(2\pi - \theta)\overset{\curvearrowright}{}[-1]](4\pi\theta - \theta^2)^{1/2}$$

$$+(2\pi - \theta)^2\left[\frac{1}{2}(4\pi\theta - \theta^2)^{-1/2}\overset{\curvearrowright}{}[4\pi - 2\theta]\right]\Bigg)$$

Exercise 4.6.42 (continued 2)

Solution (continued).

$$= \frac{1}{24\pi^2}(2\pi - \theta)\left(-2\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2}{\sqrt{4\pi\theta - \theta^2}}\right)$$

$$= \frac{1}{24\pi^2}(2\pi - \theta)\left(\frac{-2(4\pi\theta - \theta^2) + (2\pi - \theta)^2}{\sqrt{4\pi\theta - \theta^2}}\right)$$

$$= \frac{2\pi - \theta}{24\pi^2\sqrt{4\pi\theta - \theta^2}}(-8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2)$$

$$= \frac{(2\pi - \theta)(3\theta^2 - 12\pi\theta + 4\pi^2)}{24\pi^2\sqrt{4\pi\theta - \theta^2}} = V'(\theta).$$

So we look for critical points in $[0, 2\pi]$ and notice first that the endpoints $\theta = 0$ and $\theta = 2\pi$ are both critical points. Notice that $4\pi\theta - \theta^2$ is nonnegative for $\theta \in [0, 2\pi]$ so no critical points result from the term $\sqrt{4\pi\theta - \theta^2}$.

Exercise 4.6.42 (continued 2)

Solution (continued).

$$= \frac{1}{24\pi^2}(2\pi - \theta)\left(-2\sqrt{4\pi\theta - \theta^2} + \frac{(2\pi - \theta)^2}{\sqrt{4\pi\theta - \theta^2}}\right)$$

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$$= \frac{2\pi - \theta}{24\pi^2\sqrt{4\pi\theta - \theta^2}}(-8\pi\theta + 2\theta^2 + 4\pi^2 - 4\pi\theta + \theta^2)$$

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So we look for critical points in $[0, 2\pi]$ and notice first that the endpoints $\theta = 0$ and $\theta = 2\pi$ are both critical points. Notice that $4\pi\theta - \theta^2$ is nonnegative for $\theta \in [0, 2\pi]$ so no critical points result from the term $\sqrt{4\pi\theta - \theta^2}$.

Exercise 4.6.42 (continued 3)

Solution (continued). . . .

$$V'(\theta) = \frac{(2\pi - \theta)(3\theta^2 - 12\pi\theta + 4\pi^2)}{24\pi^2\sqrt{4\pi\theta - \theta^2}}.$$

So we next set $V'(\theta) = 0$ and set the factor $(3\theta^2 - 12\pi\theta + 4\pi^2)$ equal to 0. This implies from the quadratic formula that

$$\theta = \frac{-(-12\pi) \pm \sqrt{(-12\pi)^2 - 4(3)(4\pi^2)}}{2(3)} = \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6}$$

$$= \frac{12\pi \pm \sqrt{96\pi^2}}{6} = \frac{12\pi \pm 4\sqrt{6}\pi}{6} = 2\pi \pm \frac{2\sqrt{6}}{3}\pi.$$

So the only critical point in $(0, 2\pi)$ is $\theta = 2\pi - 2\sqrt{6}\pi/3$.

Exercise 4.6.42 (continued 3)

Solution (continued). ...

$$V'(\theta) = \frac{(2\pi - \theta)(3\theta^2 - 12\pi\theta + 4\pi^2)}{24\pi^2\sqrt{4\pi\theta - \theta^2}}.$$

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So the only critical point in $(0, 2\pi)$ is $\theta = 2\pi - 2\sqrt{6}\pi/3$.

Exercise 4.6.42 (continued 4)

Solution (continued). Since $V(\theta) = \frac{1}{24\pi^2}(2\pi - \theta)^2(4\pi\theta - \theta^2)^{1/2}$, then we have $V(0) = 0$, $V(2\pi) = 0$, and $V\left(2\pi - \frac{2\sqrt{6}\pi}{3}\right) =$

$$\frac{1}{24\pi^2}\left(2\pi - \left(2\pi - \frac{2\sqrt{6}\pi}{3}\right)\right)^2\left(4\pi\left(2\pi - \frac{2\sqrt{6}\pi}{3}\right) - \left(2\pi - \frac{2\sqrt{6}\pi}{3}\right)^2\right)^{1/2}$$

$$= \frac{1}{24\pi^2}\left(\frac{2\sqrt{6}\pi}{3}\right)^2\left(8\pi^2 - \frac{8\sqrt{6}\pi^2}{3} - 4\pi^2 + \frac{8\sqrt{6}\pi^2}{3} - \frac{24\pi^2}{9}\right)^{1/2}$$

$$= \frac{1}{9}(4 - 24/9)^{1/2}\pi = \frac{1}{9}\sqrt{\frac{36-24}{9}}\pi = \frac{\sqrt{12}}{27}\pi = \frac{2\sqrt{3}}{27}\pi.$$

So V is a maximum of $\boxed{\frac{2\sqrt{3}}{27}\pi}$ when $\theta = 2\pi - 2\sqrt{6}\pi/3$. □

Exercise 4.6.42 (continued 4)

Solution (continued). Since $V(\theta) = \dfrac{1}{24\pi^2}(2\pi - \theta)^2(4\pi\theta - \theta^2)^{1/2}$, then we have $V(0) = 0$, $V(2\pi) = 0$, and $V\left(2\pi - \dfrac{2\sqrt{6}\pi}{3}\right) =$

$$\frac{1}{24\pi^2}\left(2\pi - \left(2\pi - \frac{2\sqrt{6}\pi}{3}\right)\right)^2\left(4\pi\left(2\pi - \frac{2\sqrt{6}\pi}{3}\right) - \left(2\pi - \frac{2\sqrt{6}\pi}{3}\right)^2\right)^{1/2}$$

$$= \frac{1}{24\pi^2}\left(\frac{2\sqrt{6}\pi}{3}\right)^2\left(8\pi^2 - \frac{8\sqrt{6}\pi^2}{3} - 4\pi^2 + \frac{8\sqrt{6}\pi^2}{3} - \frac{24\pi^2}{9}\right)^{1/2}$$

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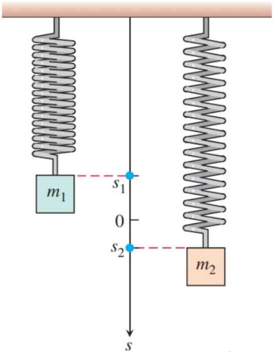
Exercise 4.6.52

Exercise 4.6.52. Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively. **(a)** At what times in the interval $t > 0$ do the masses pass each other? HINT: $\sin 2t = 2\sin t \cos t$.
(b) When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is the distance?
HINT: $\cos 2t = 2\cos^2 t - 1$.



Solution. **(a)** The masses pass each other when $s_1 = s_2$, so we consider the equation $2\sin t = \sin 2t$ or (by the hint) $2\sin t = 2\sin t \cos t$ or $2\sin t - 2\sin t\cos t = 0$ or $2\sin t(1-\cos t) = 0$. So we need either $\sin t = 0$ or $\cos t = 1$. We have $\sin t = 0$ for $t = n\pi$ where $n \in \mathbb{Z}$. We have $\cos t = 1$ for $t = 2n\pi$ where $n \in \mathbb{Z}$. So the masses pass each other for $t \in \{n\pi \mid n \in \mathbb{Z}\}$, which for $t > 0$ gives $\boxed{t \in \{n\pi \mid n \in \mathbb{N}\}}$. □

Exercise 4.6.52

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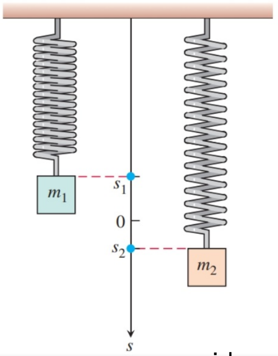

Solution. (a) The masses pass each other when $s_1 = s_2$, so we consider the equation $2\sin t = \sin 2t$ or (by the hint) $2\sin t = 2\sin t \cos t$ or $2\sin t - 2\sin t \cos t = 0$ or $2\sin t(1 - \cos t) = 0$. So we need either $\sin t = 0$ or $\cos t = 1$. We have $\sin t = 0$ for $t = n\pi$ where $n \in \mathbb{Z}$. We have $\cos t = 1$ for $t = 2n\pi$ where $n \in \mathbb{Z}$. So the masses pass each other for $t \in \{n\pi \mid n \in \mathbb{Z}\}$, which for $t > 0$ gives $\boxed{t \in \{n\pi \mid n \in \mathbb{N}\}}$. □

Exercise 4.6.52 (continued 1)

Exercise 4.6.52. Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively. **(b)** When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is the distance? HINT: $\cos 2t = 2\cos^2 t - 1$.

Solution (continued). (2 and 3) We use the picture above. The distance between the masses is $|s_1 - s_2|$.

(4) In terms of t, the distance between the masses is $|s_1 - s_2| = |2\sin t - \sin 2t|$.

Exercise 4.6.52 (continued 1)

Exercise 4.6.52. Two masses hanging side by side from springs have positions $s_1 = 2\sin t$ and $s_2 = \sin 2t$, respectively. **(b)** When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is the distance? HINT: $\cos 2t = 2\cos^2 t - 1$.
Solution (continued). (2 and 3) We use the picture above. The distance between the masses is $|s_1 - s_2|$.

(4) In terms of t, the distance between the masses is $|s_1 - s_2| = |2\sin t - \sin 2t|$.

(5) The question is: Maximize $|s_1 - s_2| = |2\sin t - \sin 2t|$ for $t \in [0, 2\pi]$. Since the absolute value function is not differentiable, then we consider the distance square $D_2(t) = |2\sin t - \sin 2t|^2 = (2\sin t - \sin 2t)^2$ (since the square function is increasing for nonnegative inputs then this will give the t values where the extrema of distance occur). We have $D_2'(t) = 2(2\sin t - \sin 2\overset{\curvearrowright}{t})[2\cos t - \cos(2\overset{\curvearrowright}{t})[2]] = 4(2\sin t - \sin 2t)(\cos t - \cos 2t)$.

Exercise 4.6.52 (continued 1)

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$$2(2\sin t - \sin \overset{\curvearrowright}{2t})[2\cos t - \cos(\overset{\curvearrowright}{2t})[2]] = 4(2\sin t - \sin 2t)(\cos t - \cos 2t).$$

Exercise 4.6.52 (continued 2)

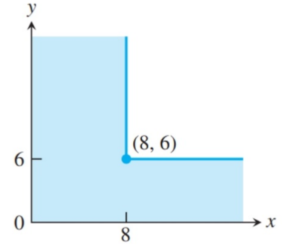
Solution (continued). $\ldots D_2'(t) = 4(2\sin t - \sin 2t)(\cos t - \cos 2t)$. We know that $2\sin t - \sin 2t = 0$ for $t \in \{0, \pi, 2\pi\}$ (since $t \in [0, 2\pi]$). We have by the hint that $\cos t - \cos 2t = \cos t - (2\cos^2 t - 1) = -2\cos^2 t + \cos t + 1 = (-2\cos t - 1)(\cos t - 1)$, so we also have critical points when $-2\cos t - 1 = 0$ or $\cos t - 1 = 0$. We have $-2\cos t - 1 = 0$, or $\cos t = -1/2$ which implies $t = 2\pi/3$ or $t = 4\pi/3$. We have $\cos t - 1 = 0$, or $\cos t = 1$ for $t = 0$ or $t = 2\pi$. So as in Section 4.1, we consider the endpoints and critical points in $[0, 2\pi]$. We have $D_2(0) = D_2(\pi) = D_2(2\pi) = (2\sin(0) - \sin(2(0)))^2 = (0)^2 = 0$, $D_2(2\pi/3) = (2\sin(2\pi/3) - \sin(2(2\pi/3)))^2 = (2(\sqrt{3}/2) - (-\sqrt{3}/2))^2 = (3\sqrt{3}/2)^2$, and. $D_2(4\pi/3) = (2\sin(4\pi/3) - \sin(2(4\pi/3)))^2 = (2(-\sqrt{3}/2) - (\sqrt{3}/2))^2 = (-3\sqrt{3}/2)^2$ So D_2 has a maximum at $\theta = 2\pi/3$ and at $\theta = 4\pi/3$. Hence the distance is greatest when $\boxed{\theta = 2\pi/3 \text{ or } \theta = 4\pi/3}$ and that distance is $\boxed{\sqrt{D_2(2\pi/3)} = \sqrt{D_2(4\pi/3)} = 3\sqrt{3}/2}$. □

Exercise 4.6.52 (continued 2)

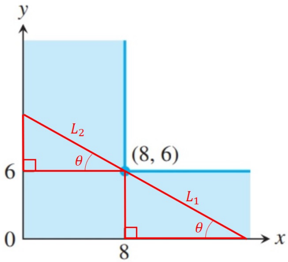
Solution (continued). $\ldots D_2'(t) = 4(2\sin t - \sin 2t)(\cos t - \cos 2t)$. We know that $2\sin t - \sin 2t = 0$ for $t \in \{0, \pi, 2\pi\}$ (since $t \in [0, 2\pi]$). We have by the hint that $\cos t - \cos 2t = \cos t - (2\cos^2 t - 1) = -2\cos^2 t + \cos t + 1 = (-2\cos t - 1)(\cos t - 1)$, so we also have critical points when $-2\cos t - 1 = 0$ or $\cos t - 1 = 0$. We have $-2\cos t - 1 = 0$, or $\cos t = -1/2$ which implies $t = 2\pi/3$ or $t = 4\pi/3$. We have $\cos t - 1 = 0$, or $\cos t = 1$ for $t = 0$ or $t = 2\pi$. So as in Section 4.1, we consider the endpoints and critical points in $[0, 2\pi]$. We have $D_2(0) = D_2(\pi) = D_2(2\pi) = (2\sin(0) - \sin(2(0)))^2 = (0)^2 = 0$, $D_2(2\pi/3) = (2\sin(2\pi/3) - \sin(2(2\pi/3)))^2 = (2(\sqrt{3}/2) - (-\sqrt{3}/2))^2 = (3\sqrt{3}/2)^2$, and. $D_2(4\pi/3) = (2\sin(4\pi/3) - \sin(2(4\pi/3)))^2 = (2(-\sqrt{3}/2) - (\sqrt{3}/2))^2 = (-3\sqrt{3}/2)^2$ So D_2 has a maximum at $\theta = 2\pi/3$ and at $\theta = 4\pi/3$. Hence the distance is greatest when $\boxed{\theta = 2\pi/3 \text{ or } \theta = 4\pi/3}$ and that distance is $\boxed{\sqrt{D_2(2\pi/3)} = \sqrt{D_2(4\pi/3)} = 3\sqrt{3}/2}$. □

Chapter 4 Practice Exercise 112

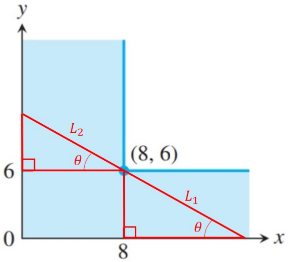
Chapter 4 Practice Exercise 112.
The Ladder Problem.
What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



HINT: Write the length of a ladder that is wedged into the hallway as $L = L_1 + L_2$ and introduce an angle θ as shown here.
Then find the minimum value of L to determine the *longest* ladder that will pass around the corner.



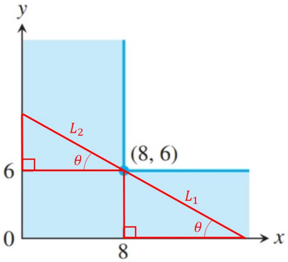
Chapter 4 Practice Exercise 112



Solution. (2 and 3) We use the picture above. Notice that $\theta \in (0, \pi/2)$. We have $L = L_1 + L_2$, $\sin\theta = 6/L_1$, and $\cos\theta = 8/L_2$. So $L_1 = 6/\sin\theta$ and $L_2 = 8/\cos\theta$.

(4) We have $L = L_1 + L_2$, so that from (3) $L(\theta) = 6/\sin\theta + 8/\cos\theta = 6\csc\theta + 8\sec\theta$.

Chapter 4 Practice Exercise 112

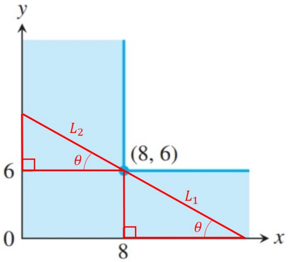


Solution. (2 and 3) We use the picture above. Notice that $\theta \in (0, \pi/2)$. We have $L = L_1 + L_2$, $\sin\theta = 6/L_1$, and $\cos\theta = 8/L_2$. So $L_1 = 6/\sin\theta$ and $L_2 = 8/\cos\theta$.

(4) We have $L = L_1 + L_2$, so that from (3)
$L(\theta) = 6/\sin\theta + 8/\cos\theta = 6\csc\theta + 8\sec\theta$.

(5) The question is: Find the minimum of $L(\theta)$ for $\theta \in (0, \pi/2)$.

Chapter 4 Practice Exercise 112



Solution. (2 and 3) We use the picture above. Notice that $\theta \in (0, \pi/2)$. We have $L = L_1 + L_2$, $\sin\theta = 6/L_1$, and $\cos\theta = 8/L_2$. So $L_1 = 6/\sin\theta$ and $L_2 = 8/\cos\theta$.

(4) We have $L = L_1 + L_2$, so that from (3)
$L(\theta) = 6/\sin\theta + 8/\cos\theta = 6\csc\theta + 8\sec\theta$.

(5) The question is: Find the minimum of $L(\theta)$ for $\theta \in (0, \pi/2)$.

Chapter 4 Practice Exercise 112 (continued 1)

Solution (continued). Since $L(\theta) = 6\csc\theta + 8\sec\theta$ then

$$L'(\theta) = -6\csc\theta\cot\theta + 8\sec\theta\tan\theta = -6\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} + 8\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}$$

$$= \frac{-6\cos\theta}{\sin^2\theta} + \frac{8\sin\theta}{\cos^2\theta} = \frac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Since we have $\theta \in (0, \pi/2)$, there are no critical points in this interval for which L' is undefined. So $L'(\theta) = 0$ implies $-6\cos^3\theta + 8\sin^3\theta = 0$ or $8\sin^3\theta = 6\cos^3\theta$ or $\frac{\sin^3\theta}{\cos^3\theta} = \frac{3}{4}$ or $\tan^3\theta = \frac{3}{4}$ or $\tan\theta = \sqrt[3]{3/4}$ (or $\theta = \tan^{-1}(\sqrt[3]{3/4})$).

Chapter 4 Practice Exercise 112 (continued 1)

Solution (continued). Since $L(\theta) = 6\csc\theta + 8\sec\theta$ then

$$L'(\theta) = -6\csc\theta\cot\theta + 8\sec\theta\tan\theta = -6\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} + 8\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}$$

$$= \frac{-6\cos\theta}{\sin^2\theta} + \frac{8\sin\theta}{\cos^2\theta} = \frac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Since we have $\theta \in (0, \pi/2)$, there are no critical points in this interval for which L' is undefined. So $L'(\theta) = 0$ implies $-6\cos^3\theta + 8\sin^3\theta = 0$ or $8\sin^3\theta = 6\cos^3\theta$ or $\dfrac{\sin^3\theta}{\cos^3\theta} = \dfrac{3}{4}$ or $\tan^3\theta = \dfrac{3}{4}$ or $\tan\theta = \sqrt[3]{3/4}$ (or $\theta = \tan^{-1}(\sqrt[3]{3/4})$).

Now $\tan^{-1} x$ is an increasing function, so $0 < \tan^{-1}(\sqrt[3]{3/8}) < \tan^{-1}(\sqrt[3]{3/4})$ and $\tan^{-1}(\sqrt[3]{3/4}) < \tan^{-1}(\sqrt[3]{4/4}) = \pi/4 < \pi/2$ (we are looking for test values for L').

Chapter 4 Practice Exercise 112 (continued 1)

Solution (continued). Since $L(\theta) = 6\csc\theta + 8\sec\theta$ then

$$L'(\theta) = -6\csc\theta\cot\theta + 8\sec\theta\tan\theta = -6\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta} + 8\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}$$

$$= \frac{-6\cos\theta}{\sin^2\theta} + \frac{8\sin\theta}{\cos^2\theta} = \frac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Since we have $\theta \in (0, \pi/2)$, there are no critical points in this interval for which L' is undefined. So $L'(\theta) = 0$ implies $-6\cos^3\theta + 8\sin^3\theta = 0$ or $8\sin^3\theta = 6\cos^3\theta$ or $\dfrac{\sin^3\theta}{\cos^3\theta} = \dfrac{3}{4}$ or $\tan^3\theta = \dfrac{3}{4}$ or $\tan\theta = \sqrt[3]{3/4}$ (or $\theta = \tan^{-1}(\sqrt[3]{3/4})$).

Now $\tan^{-1} x$ is an increasing function, so
$0 < \tan^{-1}(\sqrt[3]{3/8}) < \tan^{-1}(\sqrt[3]{3/4})$ and
$\tan^{-1}(\sqrt[3]{3/4}) < \tan^{-1}(\sqrt[3]{4/4}) = \pi/4 < \pi/2$ (we are looking for test values for L').

Chapter 4 Practice Exercise 112 (continued 2)

Solution (continued). We have $L'(\theta) = \dfrac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}$ (and the denominator is positive for all $\theta \in (0, \pi/2)$). With $\theta_k = \tan^{-1}(\sqrt[3]{3/8})$ we have $\tan\theta_k = \sqrt[3]{3/8}$ or $\tan^3\theta_k = 3/8$ or $\sin\theta_k/\cos\theta_k = 3/8$ or $\sin\theta_k = (3/8)\cos\theta_k$, and hence

$$-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3\theta_k + 8((3/8)\cos\theta_k)^3 = (-357/64)\cos^3\theta_k < 0.$$

So $L'(\tan^{-1}(\sqrt[3]{3/8})) < 0$.

Chapter 4 Practice Exercise 112 (continued 2)

Solution (continued). We have $L'(\theta) = \dfrac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}$ (and the denominator is positive for all $\theta \in (0, \pi/2)$). With $\theta_k = \tan^{-1}(\sqrt[3]{3/8})$ we have $\tan\theta_k = \sqrt[3]{3/8}$ or $\tan^3\theta_k = 3/8$ or $\sin\theta_k/\cos\theta_k = 3/8$ or $\sin\theta_k = (3/8)\cos\theta_k$, and hence

$$-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3\theta_k + 8((3/8)\cos\theta_k)^3 = (-357/64)\cos^3\theta_k < 0.$$

So $L'(\tan^{-1}(\sqrt[3]{3/8})) < 0$. With $\theta_k = \pi/4$ we have $-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3(\pi/4) + 8\sin^3(\pi/4) = -6(\sqrt{2}/2)^3 + 8(\sqrt{2}/2)^3 = 2(\sqrt{2}/2)^3 > 0$. So $L'(\pi/4) > 0$.

Chapter 4 Practice Exercise 112 (continued 2)

Solution (continued). We have $L'(\theta) = \dfrac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}$ (and the denominator is positive for all $\theta \in (0, \pi/2)$). With $\theta_k = \tan^{-1}(\sqrt[3]{3/8})$ we have $\tan\theta_k = \sqrt[3]{3/8}$ or $\tan^3\theta_k = 3/8$ or $\sin\theta_k/\cos\theta_k = 3/8$ or $\sin\theta_k = (3/8)\cos\theta_k$, and hence

$$-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3\theta_k + 8((3/8)\cos\theta_k)^3 = (-357/64)\cos^3\theta_k < 0.$$

So $L'(\tan^{-1}(\sqrt[3]{3/8})) < 0$. With $\theta_k = \pi/4$ we have $-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3(\pi/4) + 8\sin^3(\pi/4) = -6(\sqrt{2}/2)^3 + 8(\sqrt{2}/2)^3 = 2(\sqrt{2}/2)^3 > 0$. So $L'(\pi/4) > 0$. We now test the sign of L' as follows:

interval	$(0, \tan^{-1}(\sqrt[3]{3/4}))$	$(\tan^{-1}(\sqrt[3]{3/4}), \pi/2)$
test value θ_k	$\tan^{-1}(\sqrt[3]{3/8})$	$\pi/4$
$L'(\theta)$	$-$	$+$
$L(\theta)$	DEC	INC

Chapter 4 Practice Exercise 112 (continued 2)

Solution (continued). We have $L'(\theta) = \dfrac{-6\cos^3\theta + 8\sin^3\theta}{\sin^2\theta\cos^2\theta}$ (and the denominator is positive for all $\theta \in (0, \pi/2)$). With $\theta_k = \tan^{-1}(\sqrt[3]{3/8})$ we have $\tan\theta_k = \sqrt[3]{3/8}$ or $\tan^3\theta_k = 3/8$ or $\sin\theta_k/\cos\theta_k = 3/8$ or $\sin\theta_k = (3/8)\cos\theta_k$, and hence

$$-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3\theta_k + 8((3/8)\cos\theta_k)^3 = (-357/64)\cos^3\theta_k < 0.$$

So $L'(\tan^{-1}(\sqrt[3]{3/8})) < 0$. With $\theta_k = \pi/4$ we have $-6\cos^3\theta_k + 8\sin^3\theta_k = -6\cos^3(\pi/4) + 8\sin^3(\pi/4) = -6(\sqrt{2}/2)^3 + 8(\sqrt{2}/2)^3 = 2(\sqrt{2}/2)^3 > 0$. So $L'(\pi/4) > 0$. We now test the sign of L' as follows:

interval	$(0, \tan^{-1}(\sqrt[3]{3/4}))$	$(\tan^{-1}(\sqrt[3]{3/4}), \pi/2)$
test value θ_k	$\tan^{-1}(\sqrt[3]{3/8})$	$\pi/4$
$L'(\theta)$	$-$	$+$
$L(\theta)$	DEC	INC

Chapter 4 Practice Exercise 112 (continued 3)

Solution (continued). So by the First Derivative Test for Local Extrema (Theorem 4.3.A), L has a local minimum at $\theta = \tan^{-1}(\sqrt[3]{3/4})$.

Since $1 + \tan^2\theta = \sec^2\theta$ then $1 + (\sqrt[3]{3/4})^2 = \sec^2\theta$ or

$$\cos^2\theta = \frac{1}{1 + \sqrt[3]{9/16}} = \frac{\sqrt[3]{16}}{\sqrt[3]{16} + \sqrt[3]{9}} = \frac{2\sqrt[3]{2}}{2\sqrt[3]{2} + \sqrt[3]{9}}$$

or $\cos\theta = \sqrt{\dfrac{2\sqrt[3]{2}}{2\sqrt[3]{2} + \sqrt[3]{9}}}$ (notice that θ is acute so $\cos\theta > 0$).

Chapter 4 Practice Exercise 112 (continued 3)

Solution (continued). So by the First Derivative Test for Local Extrema (Theorem 4.3.A), L has a local minimum at $\theta = \tan^{-1}(\sqrt[3]{3/4})$.

Since $1 + \tan^2\theta = \sec^2\theta$ then $1 + (\sqrt[3]{3/4})^2 = \sec^2\theta$ or

$$\cos^2\theta = \frac{1}{1 + \sqrt[3]{9/16}} = \frac{\sqrt[3]{16}}{\sqrt[3]{16} + \sqrt[3]{9}} = \frac{2\sqrt[3]{2}}{2\sqrt[3]{2} + \sqrt[3]{9}}$$

or $\cos\theta = \sqrt{\dfrac{2\sqrt[3]{2}}{2\sqrt[3]{2} + \sqrt[3]{9}}}$ (notice that θ is acute so $\cos\theta > 0$).

Chapter 4 Practice Exercise 112 (continued 4)

Solution (continued). Since $\sin^2\theta = 1 - \cos^2\theta$ then

$$\sin^2\theta = 1 - \left(\sqrt{\frac{2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}}}\right)^2 = 1 - \frac{2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}}$$

$$= \frac{(2\sqrt[3]{2}+\sqrt[3]{9}) - 2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}} = \frac{\sqrt[3]{9}}{2\sqrt[3]{2}+\sqrt[3]{9}} \text{ or } \sin\theta = \sqrt{\frac{\sqrt[3]{9}}{2\sqrt[3]{2}+\sqrt[3]{9}}}.$$

So at the critical point $\theta = \tan^{-1}(\sqrt[3]{3/4})$,

$$L(\theta) = \frac{6}{\sin\theta} + \frac{8}{\cos\theta} = 6\sqrt{\frac{2\sqrt[3]{2}+\sqrt[3]{9}}{\sqrt[3]{9}}} + 8\sqrt{\frac{2\sqrt[3]{2}+\sqrt[3]{9}}{2\sqrt[3]{2}}}$$

$$= \sqrt{2\sqrt[3]{2/9}+1} + 8\sqrt{1+(1/2)\sqrt[3]{9/2}} \approx 19.7313 \text{ ft.}$$

Rounding down to the nearest foot, the ladder can be up to $\boxed{\text{19 ft long}}$. □

Chapter 4 Practice Exercise 112 (continued 4)

Solution (continued). Since $\sin^2\theta = 1 - \cos^2\theta$ then

$$\sin^2\theta = 1 - \left(\sqrt{\frac{2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}}}\right)^2 = 1 - \frac{2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}}$$

$$= \frac{(2\sqrt[3]{2}+\sqrt[3]{9}) - 2\sqrt[3]{2}}{2\sqrt[3]{2}+\sqrt[3]{9}} = \frac{\sqrt[3]{9}}{2\sqrt[3]{2}+\sqrt[3]{9}} \text{ or } \sin\theta = \sqrt{\frac{\sqrt[3]{9}}{2\sqrt[3]{2}+\sqrt[3]{9}}}.$$

So at the critical point $\theta = \tan^{-1}(\sqrt[3]{3/4})$,

$$L(\theta) = \frac{6}{\sin\theta} + \frac{8}{\cos\theta} = 6\sqrt{\frac{2\sqrt[3]{2}+\sqrt[3]{9}}{\sqrt[3]{9}}} + 8\sqrt{\frac{2\sqrt[3]{2}+\sqrt[3]{9}}{2\sqrt[3]{2}}}$$

$$= \sqrt{2\sqrt[3]{2/9}+1} + 8\sqrt{1+(1/2)\sqrt[3]{9/2}} \approx 19.7313 \text{ ft}.$$

Rounding down to the nearest foot, the ladder can be up to $\boxed{\text{19 ft long}}$. □

Exercise 4.6.62

Exercise 4.6.62. Production Level.
Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost $c(x)/x$ of making x items.

Solution. (2 and 3) There is no picture and all variables are stated in the question.

Exercise 4.6.62

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Solution. (2 and 3) There is no picture and all variables are stated in the question.

(4) The average cost function is
$A(x) = c(x)/x = (x^3 - 20x^2 + 20{,}000x)/x = x^2 - 20x + 20{,}000.$

Exercise 4.6.62

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(4) The average cost function is
$A(x) = c(x)/x = (x^3 - 20x^2 + 20{,}000x)/x = x^2 - 20x + 20{,}000.$

(5) The question is: Minimize $A(x) = c(x)/x$ for $x \in (0, \infty)$. We have $A'(x) = 2x - 20$, so that $x = 10$ is the only critical point. Notice that $A''(x) = 2$ and, in particular, $A''(10) = 2 > 0$ so that by the Second Derivative Test for Local Extrema (Theorem 4.5) A has a local minimum at $x = 10$. Since A has no other critical points, then $x = 10$ is an absolute minimum. A production level of $\boxed{\text{10 items}}$ will minimize average cost. □

Exercise 4.6.62

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$A(x) = c(x)/x = (x^3 - 20x^2 + 20{,}000x)/x = x^2 - 20x + 20{,}000.$

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