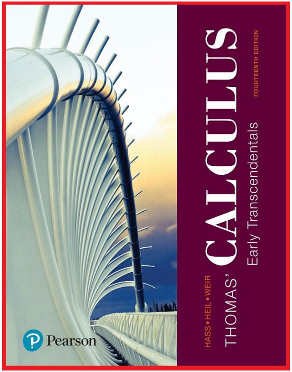
Calculus 1

Chapter 4. Applications of Derivatives

4.7. Newton's Method—Examples and Proofs



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Exercise 4.7.2. Use Newton's Method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.

Solution. First, we set $f(x) = x^3 + 3x + 1$ so that we can apply Newton's Method to the equation $f(x) = 0$, as needed. We then have $f'(x) = 3x^2 + 3$. We make a table of relevant values, as the text does in its Example 4.7.2:

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n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - f(x_n)/f'(x_n)$
0	0	$(0)^3 + 3(0) + 1 = 1$	$3(0)^2 + 3 = 3$	$(0) - (1)/(3) = -1/3$
1	$-1/3$	$(-1/3)^3 + 3(-1/3) + 1$ $= -1/27$	$3(-1/3)^2 + 3$ $= 10/3$	$(-1/3) - (-1/27)/(10/3)$ $= -29/90$

So two applications of Newton's Method produces $\boxed{x_2 = -29/90}$. □

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So two applications of Newton's Method produces $\boxed{x_2 = -29/90}$. □

Exercise 4.7.16

Exercise 4.7.16. Locating a Planet.
To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5\sin x$. Graphing the function $f(x) = x - 1 - 0.5\sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's Method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to five decimal places.)

Solution. Notice that the equation $f(x) = x - 1 - 0.5\sin x = 0$ is equivalent to the desired equation $x = 1 + 0.5\sin x$. With $f(x) = x - 1 - 0.5\sin x$, we have $f'(x) = 1 - 0.5\cos x$. With $x_0 = 1.5$ we have $f(x_0) = f(1.5) = (1.5) - 1 - 0.5\sin(1.5) = 0.5 - 0.5\sin(1.5)$ and $f'(x_0) = f'(1.5) = 1 - 0.5\cos(1.5)$.

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$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \boxed{1.5 - \frac{0.5 - 0.5\sin(1.5)}{1 - \cos(1.5)}} \approx 1.498652.$$

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Notice that if we round our approximation to four decimal places, then it agrees with the "correct" answer to four decimal places (but not to five decimal places). □

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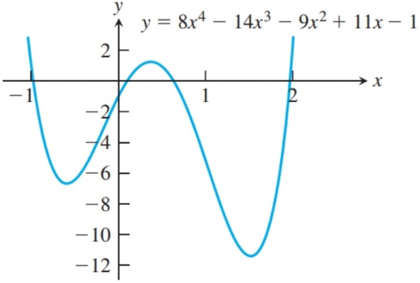
Exercise 4.7.30

Exercise 4.7.30. Factoring a Quartic.
Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



Solution. Let $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$ so that finding a root is the same as finding a solution to the equation $f(x) = 0$.

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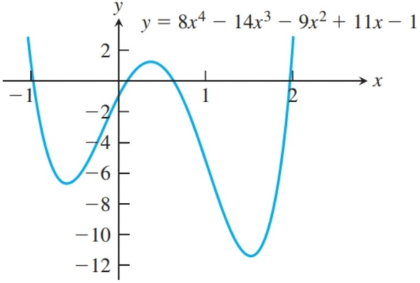

Solution. Let $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$ so that finding a root is the same as finding a solution to the equation $f(x) = 0$.

Exercise 4.7.30 (continued 1)

Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. It looks like a solution is given by $x = -1$. So to find the root close to -1, we start with $x_0 = -1$. We make a table, but we are forced to use numerical approximations (rounded to five decimal places) instead of exact values . . . ☹

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n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	$8(-1)^4 - 14(-1)^3 - 9(-1)^2$ $+11(-1) - 1 = 1$	$32(-1)^3 - 42(-1)^2$ $-18(-1) + 11 = -45$	$(-1) - \frac{(1)}{(-45)}$ $= \frac{-44}{45} \approx -0.97778$
1	$\frac{-44}{45}$	$8(\frac{-44}{45})^4 - 14(\frac{-44}{45})^3 - 9(\frac{-44}{45})^2$ $+11(\frac{-44}{45}) - 1 \approx 0.03950$	$32(\frac{-44}{45})^3 - 42(\frac{-44}{45})^2$ $-18(\frac{-44}{45}) + 11 \approx -41.46780$	$\frac{-44}{45} - \frac{0.03950}{-41.46780}$ ≈ -0.97683

We have $x_2 \approx -0.97683$ and so take the first root as $r_1 = -0.97683$.

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Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. It looks like a solution is given by $x = -1$. So to find the root close to -1, we start with $x_0 = -1$. We make a table, but we are forced to use numerical approximations (rounded to five decimal places) instead of exact values ... ☹

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	$8(-1)^4 - 14(-1)^3 - 9(-1)^2$ $+11(-1) - 1 = 1$	$32(-1)^3 - 42(-1)^2$ $-18(-1) + 11 = -45$	$(-1) - \frac{(1)}{(-45)}$ $= \frac{-44}{45} \approx -0.97778$
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Exercise 4.7.30 (continued 2)

Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. It looks like a solution is given by $x = 2$. So to find the root close to 2, we start with $x_0 = 2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	2	$8(2)^4 - 14(2)^3 - 9(2)^2$ $+11(2) - 1 = 1$	$32(2)^3 - 42(2)^2$ $-18(2) + 11 = 63$	$(2) - \frac{(1)}{(63)}$ $= \frac{125}{63} \approx 1.98412$
1	$\frac{125}{63}$	$8(\frac{125}{63})^4 - 14(\frac{125}{63})^3 - 9(\frac{125}{63})^2$ $+11(\frac{125}{63}) - 1 \approx 0.02474$	$32(\frac{125}{63})^3 - 42(\frac{125}{63})^2$ $-18(\frac{125}{63}) + 11 \approx 59.89481$	$\frac{125}{63} - \frac{0.02474}{59.89481}$ ≈ 1.98371

We have $x_2 \approx 1.98371$ and so take the root as $r_4 = 1.98371$.

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We have $x_2 \approx 1.98371$ and so take the root as $r_4 = 1.98371$.

Exercise 4.7.30 (continued 3)

Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. We start with $x_0 = 0$ and look for the root which is a bit greater than 0.

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	0	$8(0)^4 - 14(0)^3 - 9(0)^2$ $+11(0) - 1 = -1$	$32(0)^3 - 42(0)^2$ $-18(0) + 11 = 11$	$(0) - \frac{(-1)}{(11)}$ $= \frac{1}{11} \approx 0.09091$
1	$\frac{1}{11}$	$8(\frac{1}{11})^4 - 14(\frac{1}{11})^3 - 9(\frac{1}{11})^2$ $+11(\frac{1}{11}) - 1 \approx -0.08435$	$32(\frac{1}{1})^3 - 42(\frac{1}{11})^2$ $-18(\frac{1}{11}) + 11 \approx 9.01057$	$\frac{1}{11} - \frac{-0.08435}{9.01057}$ ≈ 0.10026

We have $x_2 \approx 0.10026$ and so take the root as $r_2 = 0.10026$.

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n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
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We have $x_2 \approx 0.10026$ and so take the root as $r_2 = 0.10026$.

Exercise 4.7.30 (continued 4)

Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. We start with $x_0 = 1$ and look for the root which is a bit less than 1 (we iterate Newton's Method more here since our guess is more crude).

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	$8(1)^4 - 14(1)^3 - 9(1)^2$ $+11(1) - 1 = -5$	$32(1)^3 - 42(1)^2$ $-18(1) + 11 = -17$	$(1) - \frac{(-5)}{(-17)}$ $= \frac{12}{17} \approx 0.70588$
1	$\frac{12}{17}$	$8(\frac{12}{17})^4 - 14(\frac{12}{17})^3 - 9(\frac{12}{17})^2$ $+11(\frac{12}{17}) - 1 \approx -0.65762$	$32(\frac{12}{17})^3 - 42(\frac{12}{17})^2$ $-18(\frac{12}{17}) + 11 \approx -11.37818$	$\frac{12}{17} - \frac{-0.65762}{-11.37818}$ ≈ 0.64809
2	0.64809	$8(0.64809)^4 - 14(0.64809)^3 - 9(0.64809)^2$ $+11(0.64809) - 1 \approx -0.05081$	$32(0.64809)^3 - 42(0.64809)^2$ $-18(0.64809) + 11 \approx -9.59573$	$0.64809 - \frac{-0.05081}{-9.59573}$ ≈ 0.64280

We have $x_3 \approx 0.64280$ and so take the root as $r_3 = 0.64280$.

Exercise 4.7.30 (continued 4)

Solution (continued). With $f(x) = 8x^4 - 14x^3 - 9x^2 + 11x - 1$, we have $f'(x) = 32x^3 - 42x^2 - 18x + 11$. We start with $x_0 = 1$ and look for the root which is a bit less than 1 (we iterate Newton's Method more here since our guess is more crude).

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	1	$8(1)^4 - 14(1)^3 - 9(1)^2$ $+11(1) - 1 = -5$	$32(1)^3 - 42(1)^2$ $-18(1) + 11 = -17$	$(1) - \frac{(-5)}{(-17)}$ $= \frac{12}{17} \approx 0.70588$
1	$\frac{12}{17}$	$8(\frac{12}{17})^4 - 14(\frac{12}{17})^3 - 9(\frac{12}{17})^2$ $+11(\frac{12}{17}) - 1 \approx -0.65762$	$32(\frac{12}{17})^3 - 42(\frac{12}{17})^2$ $-18(\frac{12}{17}) + 11 \approx -11.37818$	$\frac{12}{17} - \frac{-0.65762}{-11.37818}$ ≈ 0.64809
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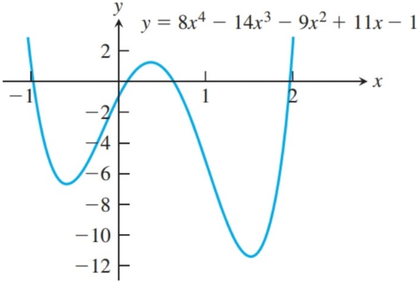
We have $x_3 \approx 0.64280$ and so take the root as $r_3 = 0.64280$.

Exercise 4.7.30 (continued 5)



Solution (continued). We approximate the roots as $\boxed{r_1 = -0.97683}$, $\boxed{r_2 = 0.10026}$, $\boxed{r_3 = 0.64280}$, and $\boxed{r_4 = 1.98371}$. $\square$

Note. A computer algebra system, such as Wolfram Alpha, gives the values to five decimal places as $r_1 = -0.97682$, $r_2 = 0.10036$, $r_3 = 0.64274$, and $r_4 = 1.98371$.

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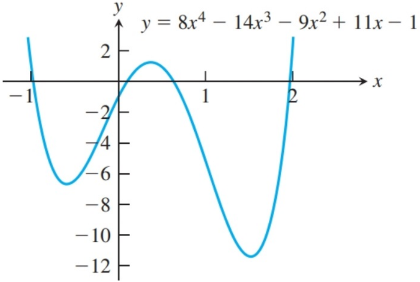

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Exercise 4.7.20

Exercise 4.7.20. Intersecting Curves.
(a) Does $\cos 3x$ ever equal x? Give reasons for your answer. **(b)** Use Newton's Method to find where.

Solution. A very careful graph of $y = \cos 3x$ and $y = x$ suggests that there are three points of intersection.

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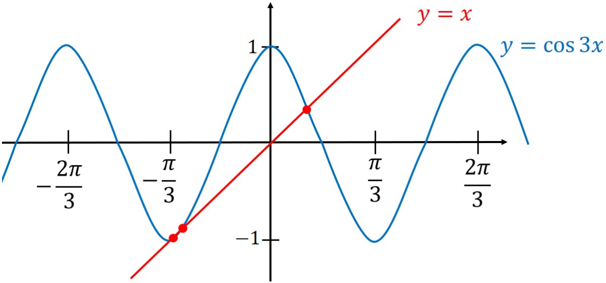

We show that there is a solution in $[0, \pi/3]$.

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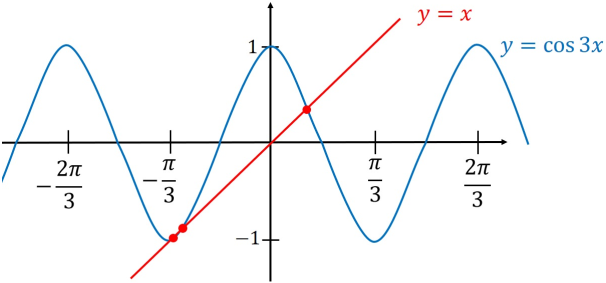

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Solution (continued). We consider $f(x) = \cos(3x) - x$ so that the equation $\cos 3x = x$ is equivalent to the equation $f(x) = 0$.

(a) We have $f(0) = \cos(3(0)) - (0) = 1 > 0$ and $f(\pi/3) = \cos(3(\pi/3)) - (\pi/3) = -1 - \pi/3 < 0$. Since $f(x) = \cos(3x) - x$ is continuous, $f(0) > 0$, and $f(\pi/3) < 0$ then by the Intermediate Value Theorem (Theorem 2.11), $\boxed{\text{YES}}$, there exists $r \in [0, \pi/3]$ such that $f(r) = 0$. □

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Solution (continued). We consider $f(x) = \cos(3x) - x$ so that the equation $\cos 3x = x$ is equivalent to the equation $f(x) = 0$.

(a) We have $f(0) = \cos(3(0)) - (0) = 1 > 0$ and $f(\pi/3) = \cos(3(\pi/3)) - (\pi/3) = -1 - \pi/3 < 0$. Since $f(x) = \cos(3x) - x$ is continuous, $f(0) > 0$, and $f(\pi/3) < 0$ then by the Intermediate Value Theorem (Theorem 2.11), $\boxed{\text{YES}}$, there exists $r \in [0, \pi/3]$ such that $f(r) = 0$. □

(b) Based on the graph, we start with $x_0 = \pi/6$. We have $f(x) = \cos(3x) - x$ then $f'(x) = -3\sin(3x) - 1$. We make a table, but we are forced to use numerical approximations (rounded to five decimal places) instead of exact values . . .

Exercise 4.7.20 (continued 1)

Exercise 4.7.20. Intersecting Curves.
(a) Does $\cos 3x$ ever equal x? Give reasons for your answer. **(b)** Use Newton's Method to find where.

Solution (continued). We consider $f(x) = \cos(3x) - x$ so that the equation $\cos 3x = x$ is equivalent to the equation $f(x) = 0$.

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(b) Based on the graph, we start with $x_0 = \pi/6$. We have $f(x) = \cos(3x) - x$ then $f'(x) = -3\sin(3x) - 1$. We make a table, but we are forced to use numerical approximations (rounded to five decimal places) instead of exact values . . . ☹

Exercise 4.7.20 (continued 1)

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Exercise 4.7.20 (continued 2)

Solution (continued).

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - f(x_n)/f'(x_n)$
0	$\pi/6$ ≈ 0.52360	$\cos(3(\pi/6)) - (\pi/6)$ $= -\pi/6 \approx -0.52360$	$-3\sin(3(\pi/6)) - 1$ $= -4$	$(\pi/6) - \frac{\cos(\pi/2)-(\pi/6)}{-3\sin(\pi/2)-1}$ ≈ 0.39270
1	0.39270	$\cos(3(0.39270)) - (0.39270)$ ≈ -0.01002	$-3\sin(3(0.39270)) - 1$ ≈ -3.77164	$(0.39270) - \frac{\cos(3(0.39270))-(0.39270)}{-3\sin(3(0.39270))-1}$ $\approx (0.39270) - \frac{-0.01002}{-3.77164} \approx 0.39004$
2	0.39004	$\cos(3(0.39004)) - (0.39004)$ ≈ 0.000001	$-3\sin(3(0.39004))$ ≈ -3.76239	$(0.39004) - \frac{\cos(3(0.39004))-(0.39004)}{-3\sin(3(0.39004))}$ $\approx (0.39004) - \frac{0.000001}{-3.76239} \approx 0.39004$

Notice that $f(x_2) \approx f(x_3) = f(0.39004) \approx 0.000001$ and to five decimal places x_2 and x_3 are the same. So we approximate the solution of $\cos 3x = x$ for $x \in [0, \pi/3]$ as $\boxed{0.39004}$. □

Exercise 4.7.20 (continued 2)

Solution (continued).

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - f(x_n)/f'(x_n)$
0	$\pi/6$ ≈ 0.52360	$\cos(3(\pi/6)) - (\pi/6)$ $= -\pi/6 \approx -0.52360$	$-3\sin(3(\pi/6)) - 1$ $= -4$	$(\pi/6) - \frac{\cos(\pi/2)-(\pi/6)}{-3\sin(\pi/2)-1}$ ≈ 0.39270
1	0.39270	$\cos(3(0.39270)) - (0.39270)$ ≈ -0.01002	$-3\sin(3(0.39270)) - 1$ ≈ -3.77164	$(0.39270) - \frac{\cos(3(0.39270))-(0.39270)}{-3\sin(3(0.39270))-1}$ $\approx (0.39270) - \frac{-0.01002}{-3.77164} \approx 0.39004$
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Notice that $f(x_2) \approx f(x_3) = f(0.39004) \approx 0.000001$ and to five decimal places x_2 and x_3 are the same. So we approximate the solution of $\cos 3x = x$ for $x \in [0, \pi/3]$ as $\boxed{0.39004}$. □

In fact, a computer algebra system, such as Wolfram Alpha, gives the solution to nine decimal places as 0.390040317 (so our x_2 is, in fact, accurate to four decimal places). The other two solutions to the equation are approximately -0.979367 and -0.887726.

Exercise 4.7.20 (continued 2)

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2	0.39004	$\cos(3(0.39004)) - (0.39004)$ ≈ 0.000001	$-3\sin(3(0.39004))$ ≈ -3.76239	$(0.39004) - \frac{\cos(3(0.39004))-(0.39004)}{-3\sin(3(0.39004))}$ $\approx (0.39004) - \frac{0.000001}{-3.76239} \approx 0.39004$

Notice that $f(x_2) \approx f(x_3) = f(0.39004) \approx 0.000001$ and to five decimal places x_2 and x_3 are the same. So we approximate the solution of $\cos 3x = x$ for $x \in [0, \pi/3]$ as $\boxed{0.39004}$. $\square$

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