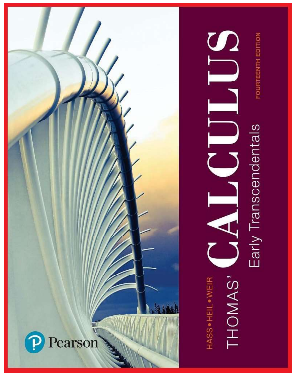
Calculus 1

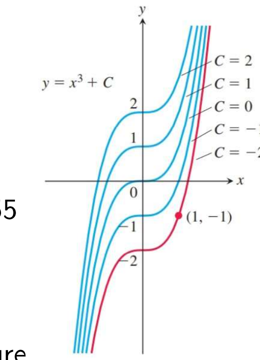
Chapter 4. Applications of Derivatives

4.8. Antiderivatives—Examples and Proofs



Example 4.8.2

Example 4.8.2. Find an antiderivative F of $f(x) = 3x^2$ that satisfies $F(1) = -1$.



Solution. By observation, an antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$. So by Theorem 4.8, the set of all antiderivatives is $\int f(x)\,dx = \int x^3\,dx = F(x) + C = x^3 + C$. So $F(x) = x^3 + k$ for some constant k. Figure 4.55 gives the graphs of such functions for various values of k. Since we require $F(1) = -1$, then we seek a value of k such that the graph of $y = F(x)$ contains the point $(1, -1)$ (as indicated in the figure in red). The condition $F(1) = -1$ implies $F(1) = (1)^3 + k = -1$ so that $1 + k = -1$ and hence $k = -2$. Therefore we have $\boxed{F(x) = x^3 - 2}$. □

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c)

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c). Find an antiderivative for each function. Check you answers by differentiation: **Exercises 4.8.2(b)** x^2, **Exercises 4.8.10(a)** $\frac{1}{2}x^{-1/2}$, **Exercises 4.8.14(b)** $\frac{\pi}{2}\cos\frac{\pi x}{2}$, **Exercises 4.8.18(b)** $4\sec 3x \tan 3x$, **Exercises 4.8.20(c)** $e^{-x/5}$.

Solutions. Exercises 4.8.2(b) x^2. An antiderivative of x^2 must involve x^3, but $\frac{d}{dx}[x^3] = 3x^2$ so we need to divide x^3 by 3 and we try $\boxed{F(x) = x^3/3}$. We check by differentiating:
$\dfrac{d}{dx}[F(x)] = \dfrac{d}{dx}\left[\dfrac{x^3}{3}\right] = \dfrac{3x^2}{3} = x^2$, so our answer is correct. □

Exercises 4.8.10(a) $\frac{1}{2}x^{-1/2}$. An antiderivative of $x^{-1/2}$ must involve $x^{-1/2+1} = x^{1/2}$, and $\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2}$. So we have $\boxed{F(x) = x^{1/2}}$. □

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c) (continued 1)

Solutions (continued). Exercises 4.8.14(b) $\frac{\pi}{2}\cos\frac{\pi x}{2}$. An antiderivative of $\cos x$ is $\sin x$, so we try $\sin\frac{\pi x}{2}$. We have $\frac{d}{dx}\left[\sin\frac{\pi x}{2}\right] = \cos\left(\frac{\pi x}{2}\right)^{\frown}\left[\frac{\pi}{2}\right]$. So we have $\boxed{F(x) = \sin\frac{\pi x}{2}}$. □

Exercises 4.8.18(b) $4\sec 3x \tan 3x$. An antiderivative of $\sec x \tan x$ is $\sec x$, so we try $\sec 3x$. We have
$\frac{d}{dx}[\sec 3x] = (\sec 3x \tan 3x)^{\frown}[3] = 3\sec 3x \tan 3x$. We need to divide out the 3 and introduce a factor of 4, so we try $\boxed{F(x) = \dfrac{4}{3}\sec 3x}$. We check by differentiating:
$\dfrac{d}{dx}[F(x)] = \dfrac{d}{dx}\left[\dfrac{4}{3}\sec 3x\right] = \dfrac{4}{3}(\sec 3x \tan 3x)^{\frown}[3] = 4\sec 3x \tan 3x$, so our answer is correct. □

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c) (continued 2)

Solutions (continued). Exercises 4.8.20(c) $e^{-x/5}$. An antiderivative of e^x is e^x, so we try $e^{-x/5}$. We have $\frac{d}{dx}\left[e^{-x/5}\right] = e^{-x/5}\left[\frac{-1}{5}\right] = \frac{-e^{-x/5}}{5}$, so we need to divide $e^{-x/5}$ by $-1/5$ (i.e., multiply by -5) and we try $\boxed{F(x) = -5e^{-x/5}}$. We check by differentiating:

$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[-5e^{-x/5}\right] = \left(-5e^{-x/5}\right)\left[\frac{-1}{5}\right] = e^{-x/5}$, so our answer is correct. □

Exercises 4.8.32

Exercises 4.8.32. Find the indefinite integral: $\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$.

Solution. We have:

$$\begin{aligned}
\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx &= \int \frac{1}{5}\, dx - \int \frac{2}{x^3}\, dx + \int 2x\, dx \\
&\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\
&= \frac{1}{5}\int 1\, dx - 2\int x^{-3}\, dx + 2\int x\, dx \\
&\quad \text{by the Constant Multiple Rule of Note 4.8.A} \\
&= \frac{1}{5}x - 2\left(\frac{x^{-3+1}}{-3+1}\right) + 2\left(\frac{x^2}{2}\right) + C \\
&\quad \text{by Table 4.2(1) with } n = 0,\ n = -3,\ \&\ n = 1 \\
&= \frac{1}{5}x + x^{-2} + x^2 + C = \boxed{\frac{1}{5}x + \frac{1}{x^2} + x^2 + C}
\end{aligned}$$ □

Exercises 4.8.46

Exercises 4.8.46. Find the indefinite integral: $\int 3\cos 5\theta\, d\theta$.

Solution. We have :

$$\begin{aligned}
\int 3\cos 5\theta\, d\theta &= 3\int \cos 5\theta\, d\theta = 3\left(\frac{\sin 5\theta}{5}\right) + C \\
&\quad \text{by Table 4.2(3) with } k = 5 \\
&= \boxed{\frac{3}{5}\sin 5\theta + C}.
\end{aligned}$$ □

Exercises 4.8.52

Exercises 4.8.52. Find the indefinite integral: $\int (2e^x - 3e^{-2x})\, dx$.

Solution. We have :

$$\begin{aligned}
\int (2e^x - 3e^{-2x})\, dx &= \int 2e^x\, dx - \int 3e^{-2x}\, dx \\
&\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\
&= 2\int e^x\, dx - 3\int e^{-2x}\, dx \\
&\quad \text{by the Constant Multiple Rule of Note 4.8.A} \\
&= 2(e^x) - 3\left(\frac{e^{-2x}}{-2}\right) + C \\
&\quad \text{by Table 4.2(8) with } k = 1 \text{ and } k = -2 \\
&= \boxed{2e^x + \frac{3}{2}e^{-2x} + C}.
\end{aligned}$$ □

Exercises 4.8.54

Exercises 4.8.54. Find the indefinite integral: $\int (1.3)^x\,dx$.

Solution. We have :

$$\begin{aligned}\int (1.3)^x\,dx &= \left(\frac{1}{\ln 1.3}\right)(1.3)^x + C \\ &\quad \text{by Table 4.2(13) with } k=1 \text{ and } a=1.3 \\ &= \boxed{\frac{(1.3)^x}{\ln 1.3} + C}. \ \square\end{aligned}$$

Exercises 4.8.66

Exercises 4.8.66. Find the indefinite integral: $\int (2+\tan^2\theta)\,d\theta$. HINT: $1+\tan^2\theta = \sec^2\theta$.

Solution. Notice that we don't (yet) know how to antidifferentiate $\tan^2\theta$, but we do know how to antidifferentiate $\sec^2\theta$, since $\frac{d}{d\theta}[\tan\theta] = \sec^2\theta$. We have

$$\begin{aligned}\int (2+\tan^2\theta)\,d\theta &= \int (2+(\sec^2\theta - 1))\,d\theta = \int (1+\sec^2\theta)\,d\theta \\ &= \int 1\,d\theta + \int \sec^2\theta\,d\theta \\ &\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\ &= \boxed{\theta + \tan\theta + C} \text{ by Table 4.2(1 and 4)} \\ &\quad \text{with } n=0 \text{ and } k=1. \ \square\end{aligned}$$

Exercises 4.8.76

Exercises 4.8.76. Verify the formula by differentiation:
$\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$.

Solution. Recall that an indefinite integral is a *set* of functions. So for $F(x) \in \int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$ we have that $F(x) = \frac{x}{x+1} + k$ for some k. Now

$$F'(x) = \frac{[1](x+1) - (x)[1]}{(x+1)^2} + 0 = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

So $F(x) = \frac{x}{x+1} + k$ is *an* antiderivative of $\frac{1}{(x+1)^2}$. By Theorem 4.8, the indefinite integral of $\frac{1}{(x+1)^2}$ is $F(x) + C = \frac{x}{x+1} + C$ (we have absorbed k in the "arbitrary constant" C), as claimed. $\square$

Exercises 4.8.94

Exercises 4.8.94. Solve the initial value problem: $\frac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$.

Solution. Let $y = f(x)$ where $\frac{dy}{dx} = \frac{df}{dx} = 9x^2 - 4x + 5$, so f is an antiderivative of $9x^2 - 4x + 5$; that is, $f(x) \in \int 9x^2 - 4x + 5\,dx$. Now

$$\int 9x^2 - 4x + 5\,dx = 9\int x^2\,dx - 4\int x\,dx + 5\int 1\,dx$$

$$= 9\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5(x) + C = 3x^3 - 2x^2 + 5x + C.$$

So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k.

Exercises 4.8.94 (continuous)

Exercises 4.8.94. Solve the initial value problem: $\dfrac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$.

Solution (continuous. So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k. We use the initial condition $y(-1) = f(-1) = 0$ to find k. We set $f(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + k = 0$ which requires $-10 + k = 0$ or $k = 10$. So $f(x) = 3x^3 - 2x^2 + 5x + (10) = \boxed{3x^3 - 2x^2 + 5x + 10}$. $\square$

Exercises 4.8.102

Exercises 4.8.102. Solve the initial value problem: $\dfrac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution. With $\dfrac{dv}{dt} = 8t + \csc^2 t$, we have that $v(t)$ is an antiderivative of $8t + \csc^2 t$; that is, $v(t) \in \int 8t + \csc^2 t\, dt$. Now

$$\int 8t + \csc^2 t\, dt = 8\int t\, dt + \int \csc^2 t\, dt$$

$$= 8\left(\frac{t^2}{2}\right) + (-\cot t) + C = 4t^2 - \cot t + C$$

by Table 4.2(1 and 5) with $n = 1$ and $k = 1$. So $v(t) = 4t^2 - \cot t + k$ for some constant k.

Exercises 4.8.102 (continued)

Exercises 4.8.102. Solve the initial value problem: $\dfrac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution (continued). So $v(t) = 4t^2 - \cot t + k$ for some constant k. We use the initial condition $v(\pi/2) = -7$ to find k. We set $v(\pi/2) = 4(\pi/2)^2 - \cot(\pi/2) + k = -7$ which requires $\pi^2 + k = -7$ or $k = -7 - \pi^2$. So $\boxed{v(t) = 4t^2 - \cot t - 7 - \pi^2}$. $\square$

Exercises 4.8.108

Exercises 4.8.108. Solve the "second order" initial value problem: $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}$, $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$, $s(4) = 4$.

Solution. We look for $s(t)$. We have $\dfrac{ds}{dt}$ is an antiderivative of $\dfrac{d^2s}{dt^2}$; that is, $\dfrac{ds}{dt} \in \int \dfrac{d^2s}{dt^2}\, dt$. Now

$$\int \frac{3t}{8}\, dt = \frac{3}{8}\int t\, dt = \frac{3}{8}\left(\frac{t^2}{2}\right) + C = \frac{3}{16}t^2 + C.$$

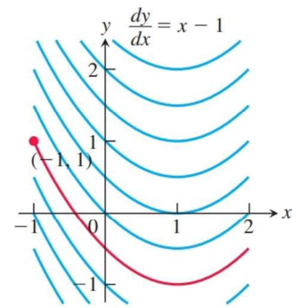
So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2 + k_1$ for some constant k_1. We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1.

Exercises 4.8.108 (continued)

Solution (continued). We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1. We have $\left.\dfrac{ds}{dt}\right|_{t=4} = \dfrac{3}{16}(4)^2 + k_1 = 3$ which requires $3 + k_1 = 3$ or $k_1 = 0$. So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$. Next, $s(t)$ is an antiderivative of $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$; that is, $s(t) \in \displaystyle\int \frac{ds}{dt}\,dt = \int \frac{3}{16}t^2\,dt$. Now $\displaystyle\int \frac{3}{16}t^2\,dt = \frac{3}{16}\frac{t^3}{3} + C = \frac{t^3}{16} + C$. So $s(t) = \dfrac{t^3}{16} + k_2$ for some constant k_2. We use the initial condition $s(4) = 4$ to find k_2. We have $s(4) = \dfrac{(4)^3}{16} + k_2 = 4$ which requires $4 + k_2 = 4$ or $k_2 = 0$. So $\boxed{s(t) = \dfrac{t^3}{16}}$. □

Exercises 4.8.120

Exercises 4.8.120. Consider the figure with solution curves of the given differential equation. Find an equation for the curve through the labeled point.



Solution. Let $y = f(x)$ where $\dfrac{dy}{dx} = \dfrac{df}{dx} = x - 1$, so f is an antiderivative of $x - 1$; that is, $f(x) \in \displaystyle\int x - 1\,dx$.

Exercises 4.8.120 (continued)

Solution (continued). Now

$$\int x - 1\,dx = \int x\,dx - \int 1\,dx = \left(\frac{x^2}{2}\right) - x + C = \frac{x^2}{2} - x + C.$$

So $f(x) = \dfrac{x^2}{2} - x + k$ for some constant k. The fact that the graph of the desired function f passes through the point $(-1, 1)$ gives us the initial condition $f(-1) = 1$. We use this initial condition to find k. We set $f(-1) = \dfrac{(-1)^2}{2} - (-1) + k = 1$ which requires $3/2 + k = 1$ or $k = -1/2$. So $\boxed{f(x) = \dfrac{x^2}{2} - x - \dfrac{1}{2}}$. □

Exercises 4.8.124

Exercises 4.8.124. Liftoff from Earth.
A rocket lifts off from the surface of the Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later.

Solution. We let $v(t)$ represent the velocity of the rocket in m/sec at time t sec after liftoff. So $v(t)$ is an antiderivative of acceleration $a(t) = 20$ m/sec^2; that is, $v(t) \in \displaystyle\int 20\,dt$. Now $\displaystyle\int 20\,dt = 20t + C$ so $v(t) = 20t + k$ for some constant k. We need an initial value to find k. Since the rocket starts stationary on the launch pad, then $v(0) = 0$ m/sec. So we set $v(0) = 20(0) + k = 0$ which requires $k = 0$. Hence $v(t) = 20t$ m/sec is the velocity function. When $t = 1$ min $= 60$ sec the velocity of the rocket is $v(60) = 20(60) = \boxed{1200 \text{ m/sec}}$. □