Calculus 1

Chapter 4. Applications of Derivatives

4.8. Antiderivatives—Examples and Proofs

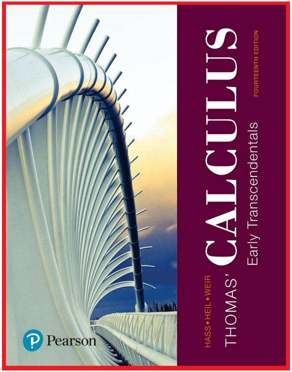

Table of contents

1. Example 4.8.2
2. Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c)
3. Exercises 4.8.32
4. Exercises 4.8.46
5. Exercises 4.8.52
6. Exercises 4.8.54
7. Exercises 4.8.66
8. Exercises 4.8.76
9. Exercises 4.8.94
10. Exercises 4.8.102
11. Exercises 4.8.108
12. Exercises 4.8.120
13. Exercises 4.8.124. Liftoff from Earth

Example 4.8.2

Example 4.8.2. Find an antiderivative F of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution. By observation, an antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$. So by Theorem 4.8, the set of all antiderivatives is $\int f(x)\,dx = \int x^3\,dx = F(x) + C = x^3 + C$. So $F(x) = x^3 + k$ for some constant k.

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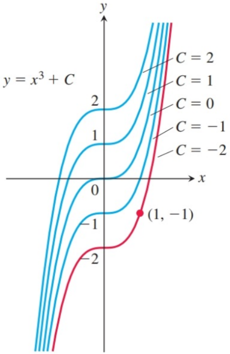

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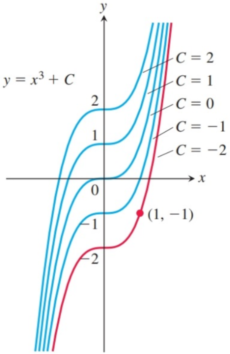

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Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c)

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c). Find an antiderivative for each function. Check you answers by differentiation: **Exercises 4.8.2(b)** x^2, **Exercises 4.8.10(a)** $\frac{1}{2}x^{-1/2}$, **Exercises 4.8.14(b)** $\frac{\pi}{2}\cos\frac{\pi x}{2}$, **Exercises 4.8.18(b)** $4\sec 3x\tan 3x$, **Exercises 4.8.20(c)** $e^{-x/5}$.

Solutions. Exercises 4.8.2(b) x^2. An antiderivative of x^2 must involve x^3, but $\frac{d}{dx}[x^3] = 3x^2$ so we need to divide x^3 by 3 and we try $\boxed{F(x) = x^3/3}$. We check by differentiating:

$\dfrac{d}{dx}[F(x)] = \dfrac{d}{dx}\left[\dfrac{x^3}{3}\right] = \dfrac{3x^2}{3} = x^2$, so our answer is correct. $\square$

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$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\frac{x^3}{3}\right] = \frac{3x^2}{3} = x^2,$$ so our answer is correct. □

Exercises 4.8.10(a) $\frac{1}{2}x^{-1/2}$. An antiderivative of $x^{-1/2}$ must involve $x^{-1/2+1} = x^{1/2}$, and $\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2}$. So we have $\boxed{F(x) = x^{1/2}}$. □

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c)

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c). Find an antiderivative for each function. Check you answers by differentiation: **Exercises 4.8.2(b)** x^2, **Exercises 4.8.10(a)** $\frac{1}{2}x^{-1/2}$, **Exercises 4.8.14(b)** $\frac{\pi}{2}\cos\frac{\pi x}{2}$, **Exercises 4.8.18(b)** $4\sec 3x \tan 3x$, **Exercises 4.8.20(c)** $e^{-x/5}$.

Solutions. Exercises 4.8.2(b) x^2. An antiderivative of x^2 must involve x^3, but $\frac{d}{dx}[x^3] = 3x^2$ so we need to divide x^3 by 3 and we try $\boxed{F(x) = x^3/3}$. We check by differentiating:
$\dfrac{d}{dx}[F(x)] = \dfrac{d}{dx}\left[\dfrac{x^3}{3}\right] = \dfrac{3x^2}{3} = x^2$, so our answer is correct. $\square$

Exercises 4.8.10(a) $\frac{1}{2}x^{-1/2}$. An antiderivative of $x^{-1/2}$ must involve $x^{-1/2+1} = x^{1/2}$, and $\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2}$. So we have $\boxed{F(x) = x^{1/2}}$. $\square$

Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c) (continued 1)

Solutions (continued). Exercises 4.8.14(b) $\frac{\pi}{2}\cos\frac{\pi x}{2}$. An antiderivative of $\cos x$ is $\sin x$, so we try $\sin\frac{\pi x}{2}$. We have $\frac{d}{dx}\left[\sin\frac{\pi x}{2}\right] = \cos\left(\frac{\pi x}{2}\right)\left[\frac{\pi}{2}\right]$. So we have $\boxed{F(x) = \sin\frac{\pi x}{2}}$. □

Exercises 4.8.18(b) $4\sec 3x\tan 3x$. An antiderivative of $\sec x\tan x$ is $\sec x$, so we try $\sec 3x$. We have

$\frac{d}{dx}[\sec 3x] = (\sec 3x\tan 3x)[3] = 3\sec 3x\tan 3x$. We need to divide out the 3 and introduce a factor of 4, so we try $\boxed{F(x) = \dfrac{4}{3}\sec 3x}$. We check by differentiating:

$\dfrac{d}{dx}[F(x)] = \dfrac{d}{dx}\left[\dfrac{4}{3}\sec 3x\right] = \dfrac{4}{3}(\sec 3x\tan 3x)[3] = 4\sec 3x\tan 3x$, so our answer is correct. □

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$\frac{d}{dx}[\sec 3x] = (\sec 3x\tan 3x)[3] = 3\sec 3x\tan 3x$. We need to divide out the 3 and introduce a factor of 4, so we try $\boxed{F(x) = \dfrac{4}{3}\sec 3x}$. We check by differentiating:
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Exercises 4.8.2(b), 4.8.10(a), 4.8.14(b), 4.8.18(b), and 4.8.20(c) (continued 2)

Solutions (continued). Exercises 4.8.20(c) $e^{-x/5}$. An antiderivative of e^x is e^x, so we try $e^{-x/5}$. We have $\frac{d}{dx}\left[e^{-x/5}\right] = e^{-x/5}\left[\frac{-1}{5}\right] = \frac{-e^{-x/5}}{5}$, so we need to divide $e^{-x/5}$ by $-1/5$ (i.e., multiply by -5) and we try $\boxed{F(x) = -5e^{-x/5}}$. We check by differentiating:

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}\left[-5e^{-x/5}\right] = \left(-5e^{-x/5}\right)\left[\frac{-1}{5}\right] = e^{-x/5},$$ so our answer is correct. □

Exercises 4.8.32

Exercises 4.8.32. Find the indefinite integral: $\displaystyle\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$.

Solution. We have:

$$\begin{aligned}
\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx &= \int \frac{1}{5}\,dx - \int \frac{2}{x^3}\,dx + \int 2x\,dx \\
&\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\
&= \frac{1}{5}\int 1\,dx - 2\int x^{-3}\,dx + 2\int x\,dx \\
&\quad \text{by the Constant Multiple Rule of Note 4.8.A} \\
&= \frac{1}{5}x - 2\left(\frac{x^{-3+1}}{-3+1}\right) + 2\left(\frac{x^2}{2}\right) + C \\
&\quad \text{by Table 4.2(1) with } n = 0,\ n = -3,\ \&\ n = 1 \\
&= \frac{1}{5}x + x^{-2} + x^2 + C = \boxed{\frac{1}{5}x + \frac{1}{x^2} + x^2 + C} \quad \square
\end{aligned}$$

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&= \frac{1}{5} x + x^{-2} + x^2 + C = \boxed{\frac{1}{5} x + \frac{1}{x^2} + x^2 + C} \quad \square
\end{aligned}$$

Exercises 4.8.46

Exercises 4.8.46. Find the indefinite integral: $\int 3\cos 5\theta \, d\theta$.

Solution. We have :

$$\begin{aligned} \int 3\cos 5\theta \, d\theta &= 3\int \cos 5\theta \, d\theta = 3\left(\frac{\sin 5\theta}{5}\right) + C \\ &\quad \text{by Table 4.2(3) with } k = 5 \\ &= \boxed{\frac{3}{5}\sin 5\theta + C}. \quad \square \end{aligned}$$

Exercises 4.8.46

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Exercises 4.8.52

Exercises 4.8.52. Find the indefinite integral: $\displaystyle\int (2e^x - 3e^{-2x})\,dx$.

Solution. We have :

$$\begin{aligned}
\int (2e^x - 3e^{-2x})\,dx &= \int 2e^x\,dx - \int 3e^{-2x}\,dx \\
&\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\
&= 2\int e^x\,dx - 3\int e^{-2x}\,dx \\
&\quad \text{by the Constant Multiple Rule of Note 4.8.A} \\
&= 2(e^x) - 3\left(\frac{e^{-2x}}{-2}\right) + C \\
&\quad \text{by Table 4.2(8) with } k = 1 \text{ and } k = -2 \\
&= \boxed{2e^x + \frac{3}{2}e^{-2x} + C}. \quad \square
\end{aligned}$$

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&= 2(e^x) - 3\left(\frac{e^{-2x}}{-2}\right) + C \\
&\quad \text{by Table 4.2(8) with } k = 1 \text{ and } k = -2 \\
&= \boxed{2e^x + \frac{3}{2}e^{-2x} + C}. \quad \square
\end{aligned}$$

Exercises 4.8.54

Exercises 4.8.54. Find the indefinite integral: $\int (1.3)^x \, dx$.

Solution. We have :

$$\int (1.3)^x \, dx = \left(\frac{1}{\ln 1.3}\right)(1.3)^x + C$$

by Table 4.2(13) with $k = 1$ and $a = 1.3$

$$= \boxed{\frac{(1.3)^x}{\ln 1.3} + C}. \quad \square$$

Exercises 4.8.54

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Solution. We have :

$$\begin{aligned}\int (1.3)^x\,dx &= \left(\frac{1}{\ln 1.3}\right)(1.3)^x + C \\ &\qquad \text{by Table 4.2(13) with } k = 1 \text{ and } a = 1.3 \\ &= \boxed{\frac{(1.3)^x}{\ln 1.3} + C}. \ \square\end{aligned}$$

Exercises 4.8.66

Exercises 4.8.66. Find the indefinite integral: $\int (2 + \tan^2 \theta)\, d\theta$. HINT: $1 + \tan^2 \theta = \sec^2 \theta$.

Solution. Notice that we don't (yet) know how to antidifferentiate $\tan^2 \theta$, but we do know how to antidifferentiate $\sec^2 \theta$, since $\frac{d}{d\theta}[\tan \theta] = \sec^2 \theta$.

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Solution. Notice that we don't (yet) know how to antidifferentiate $\tan^2\theta$, but we do know how to antidifferentiate $\sec^2\theta$, since $\frac{d}{d\theta}[\tan\theta] = \sec^2\theta$. We have

$$\begin{aligned} \int (2 + \tan^2\theta)\, d\theta &= \int (2 + (\sec^2\theta - 1))\, d\theta = \int (1 + \sec^2\theta)\, d\theta \\ &= \int 1\, d\theta + \int \sec^2\theta\, d\theta \\ &\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\ &= \boxed{\theta + \tan\theta + C} \text{ by Table 4.2(1 and 4)} \\ &\quad \text{with } n = 0 \text{ and } k = 1. \ \square \end{aligned}$$

Exercises 4.8.66

Exercises 4.8.66. Find the indefinite integral: $\displaystyle\int (2+\tan^2\theta)\,d\theta$. HINT: $1+\tan^2\theta = \sec^2\theta$.

Solution. Notice that we don't (yet) know how to antidifferentiate $\tan^2\theta$, but we do know how to antidifferentiate $\sec^2\theta$, since $\frac{d}{d\theta}[\tan\theta] = \sec^2\theta$. We have

$$\begin{aligned}
\int (2+\tan^2\theta)\,d\theta &= \int (2+(\sec^2\theta-1))\,d\theta = \int (1+\sec^2\theta)\,d\theta \\
&= \int 1\,d\theta + \int \sec^2\theta\,d\theta \\
&\quad \text{by the Sum or Difference Rules of Note 4.8.A} \\
&= \boxed{\theta+\tan\theta+C} \text{ by Table 4.2(1 and 4)} \\
&\quad \text{with } n=0 \text{ and } k=1. \quad \square
\end{aligned}$$

Exercises 4.8.76

Exercises 4.8.76. Verify the formula by differentiation:
$$\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C.$$

Solution. Recall that an indefinite integral is a *set* of functions. So for $F(x) \in \int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$ we have that $F(x) = \frac{x}{x+1} + k$ for some k.

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$$F'(x) = \frac{[1](x+1)-(x)[1]}{(x+1)^2} + 0 = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

So $F(x) = \frac{x}{x+1} + k$ is *an* antiderivative of $\frac{1}{(x+1)^2}$. By Theorem 4.8, the indefinite integral of $\frac{1}{(x+1)^2}$ is $F(x) + C = \frac{x}{x+1} + C$ (we have absorbed k in the "arbitrary constant" C), as claimed. □

Exercises 4.8.76

Exercises 4.8.76. Verify the formula by differentiation:
$\displaystyle\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C.$

Solution. Recall that an indefinite integral is a *set* of functions. So for $F(x) \in \displaystyle\int \frac{1}{(x+1)^2}\,dx = \frac{x}{x+1} + C$ we have that $F(x) = \dfrac{x}{x+1} + k$ for some k. Now

$$F'(x) = \frac{[1](x+1) - (x)[1]}{(x+1)^2} + 0 = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}.$$

So $F(x) = \dfrac{x}{x+1} + k$ is *an* antiderivative of $\dfrac{1}{(x+1)^2}$. By Theorem 4.8, the indefinite integral of $\dfrac{1}{(x+1)^2}$ is $F(x) + C = \dfrac{x}{x+1} + C$ (we have absorbed k in the "arbitrary constant" C), as claimed. □

Exercises 4.8.94

Exercises 4.8.94. Solve the initial value problem: $\dfrac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$.

Solution. Let $y = f(x)$ where $\dfrac{dy}{dx} = \dfrac{df}{dx} = 9x^2 - 4x + 5$, so f is an antiderivative of $9x^2 - 4x + 5$; that is, $f(x) \in \displaystyle\int 9x^2 - 4x + 5\,dx$. Now

$$\int 9x^2 - 4x + 5\,dx = 9\int x^2\,dx - 4\int x\,dx + 5\int 1\,dx$$

$$= 9\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5(x) + C = 3x^3 - 2x^2 + 5x + C.$$

So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k.

Exercises 4.8.94

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Solution. Let $y = f(x)$ where $\dfrac{dy}{dx} = \dfrac{df}{dx} = 9x^2 - 4x + 5$, so f is an antiderivative of $9x^2 - 4x + 5$; that is, $f(x) \in \int 9x^2 - 4x + 5\,dx$. Now

$$\int 9x^2 - 4x + 5\,dx = 9\int x^2\,dx - 4\int x\,dx + 5\int 1\,dx$$

$$= 9\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5(x) + C = 3x^3 - 2x^2 + 5x + C.$$

So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k.

Exercises 4.8.94 (continuous)

Exercises 4.8.94. Solve the initial value problem: $\dfrac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$.

Solution (continuous. So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k. We use the initial condition $y(-1) = f(-1) = 0$ to find k. We set $f(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + k = 0$ which requires $-10 + k = 0$ or $k = 10$. So $f(x) = 3x^3 - 2x^2 + 5x + (10) = \boxed{3x^3 - 2x^2 + 5x + 10}$. □

Exercises 4.8.94 (continuous)

Exercises 4.8.94. Solve the initial value problem: $\dfrac{dy}{dx} = 9x^2 - 4x + 5$, $y(-1) = 0$.

Solution (continuous. So $f(x) = 3x^3 - 2x^2 + 5x + k$ for some constant k. We use the initial condition $y(-1) = f(-1) = 0$ to find k. We set $f(-1) = 3(-1)^3 - 2(-1)^2 + 5(-1) + k = 0$ which requires $-10 + k = 0$ or $k = 10$. So $f(x) = 3x^3 - 2x^2 + 5x + (10) = \boxed{3x^3 - 2x^2 + 5x + 10}$. □

Exercises 4.8.102

Exercises 4.8.102. Solve the initial value problem: $\frac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution. With $\frac{dv}{dt} = 8t + \csc^2 t$, we have that $v(t)$ is an antiderivative of $8t + \csc^2 t$; that is, $v(t) \in \int 8t + \csc^2 t\, dt$. Now

$$\int 8t + \csc^2 t\, dt = 8\int t\, dt + \int \csc^2 t\, dt$$

$$= 8\left(\frac{t^2}{2}\right) + (-\cot t) + C = 4t^2 - \cot t + C$$

by Table 4.2(1 and 5) with $n = 1$ and $k = 1$. So $v(t) = 4t^2 - \cot t + k$ for some constant k.

Exercises 4.8.102

Exercises 4.8.102. Solve the initial value problem: $\dfrac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution. With $\dfrac{dv}{dt} = 8t + \csc^2 t$, we have that $v(t)$ is an antiderivative of $8t + \csc^2 t$; that is, $v(t) \in \displaystyle\int 8t + \csc^2 t \, dt$. Now

$$\int 8t + \csc^2 t \, dt = 8 \int t \, dt + \int \csc^2 t \, dt$$

$$= 8\left(\frac{t^2}{2}\right) + (-\cot t) + C = 4t^2 - \cot t + C$$

by Table 4.2(1 and 5) with $n = 1$ and $k = 1$. So $v(t) = 4t^2 - \cot t + k$ for some constant k.

Exercises 4.8.102 (continued)

Exercises 4.8.102. Solve the initial value problem: $\dfrac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution (continued). So $v(t) = 4t^2 - \cot t + k$ for some constant k. We use the initial condition $v(\pi/2) = -7$ to find k. We set $v(\pi/2) = 4(\pi/2)^2 - \cot(\pi/2) + k = -7$ which requires $\pi^2 + k = -7$ or $k = -7 - \pi^2$. So $\boxed{v(t) = 4t^2 - \cot t - 7 - \pi^2}$. □

Exercises 4.8.102 (continued)

Exercises 4.8.102. Solve the initial value problem: $\dfrac{dv}{dt} = 8t + \csc^2 t$, $v(\pi/2) = -7$.

Solution (continued). So $v(t) = 4t^2 - \cot t + k$ for some constant k. We use the initial condition $v(\pi/2) = -7$ to find k. We set $v(\pi/2) = 4(\pi/2)^2 - \cot(\pi/2) + k = -7$ which requires $\pi^2 + k = -7$ or $k = -7 - \pi^2$. So $\boxed{v(t) = 4t^2 - \cot t - 7 - \pi^2}$. □

Exercises 4.8.108

Exercises 4.8.108. Solve the "second order" initial value problem: $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}$, $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$, $s(4) = 4$.

Solution. We look for $s(t)$. We have $\dfrac{ds}{dt}$ is an antiderivative of $\dfrac{d^2s}{dt^2}$; that is, $\dfrac{ds}{dt} \in \displaystyle\int \frac{d^2s}{dt^2}\,dt$. Now

$$\int \frac{3t}{8}\,dt = \frac{3}{8}\int t\,dt = \frac{3}{8}\left(\frac{t^2}{2}\right) + C = \frac{3}{16}t^2 + C.$$

So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2 + k_1$ for some constant k_1. We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1.

Exercises 4.8.108

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Solution. We look for $s(t)$. We have $\dfrac{ds}{dt}$ is an antiderivative of $\dfrac{d^2s}{dt^2}$; that is, $\dfrac{ds}{dt} \in \displaystyle\int \frac{d^2s}{dt^2}\,dt$. Now

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So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2 + k_1$ for some constant k_1. We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1.

Exercises 4.8.108 (continued)

Solution (continued). We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1. We have $\left.\dfrac{ds}{dt}\right|_{t=4} = \dfrac{3}{16}(4)^2 + k_1 = 3$ which requires $3 + k_1 = 3$ or $k_1 = 0$. So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$.

Exercises 4.8.108 (continued)

Solution (continued). We use the initial condition $\left.\frac{ds}{dt}\right|_{t=4} = 3$ to find k_1. We have $\left.\frac{ds}{dt}\right|_{t=4} = \frac{3}{16}(4)^2 + k_1 = 3$ which requires $3 + k_1 = 3$ or $k_1 = 0$. So $\frac{ds}{dt} = \frac{3}{16}t^2$. Next, $s(t)$ is an antiderivative of $\frac{ds}{dt} = \frac{3}{16}t^2$; that is, $s(t) \in \int \frac{ds}{dt}\,dt = \int \frac{3}{16}t^2\,dt$. Now $\int \frac{3}{16}t^2\,dt = \frac{3}{16}\frac{t^3}{3} + C = \frac{t^3}{16} + C$. So $s(t) = \frac{t^3}{16} + k_2$ for some constant k_2. We use the initial condition $s(4) = 4$ to find k_2.

Exercises 4.8.108 (continued)

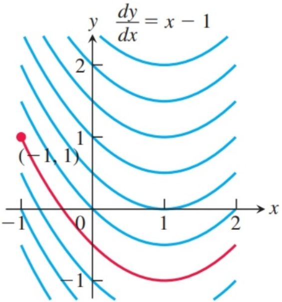
Solution (continued). We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1. We have $\left.\dfrac{ds}{dt}\right|_{t=4} = \dfrac{3}{16}(4)^2 + k_1 = 3$ which requires $3 + k_1 = 3$ or $k_1 = 0$. So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$. Next, $s(t)$ is an antiderivative of $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$; that is, $s(t) \in \displaystyle\int \dfrac{ds}{dt}\,dt = \int \dfrac{3}{16}t^2\,dt$. Now $\displaystyle\int \dfrac{3}{16}t^2\,dt = \dfrac{3}{16}\dfrac{t^3}{3} + C = \dfrac{t^3}{16} + C$. So $s(t) = \dfrac{t^3}{16} + k_2$ for some constant k_2. We use the initial condition $s(4) = 4$ to find k_2. We have $s(4) = \dfrac{(4)^3}{16} + k_2 = 4$ which requires $4 + k_2 = 4$ or $k_2 = 0$. So $\boxed{s(t) = \dfrac{t^3}{16}}$. □

Exercises 4.8.108 (continued)

Solution (continued). We use the initial condition $\left.\dfrac{ds}{dt}\right|_{t=4} = 3$ to find k_1. We have $\left.\dfrac{ds}{dt}\right|_{t=4} = \dfrac{3}{16}(4)^2 + k_1 = 3$ which requires $3 + k_1 = 3$ or $k_1 = 0$. So $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$. Next, $s(t)$ is an antiderivative of $\dfrac{ds}{dt} = \dfrac{3}{16}t^2$; that is, $s(t) \in \displaystyle\int \frac{ds}{dt}\,dt = \int \frac{3}{16}t^2\,dt$. Now $\displaystyle\int \frac{3}{16}t^2\,dt = \frac{3}{16}\frac{t^3}{3} + C = \frac{t^3}{16} + C$. So $s(t) = \dfrac{t^3}{16} + k_2$ for some constant k_2. We use the initial condition $s(4) = 4$ to find k_2. We have $s(4) = \dfrac{(4)^3}{16} + k_2 = 4$ which requires $4 + k_2 = 4$ or $k_2 = 0$. So $\boxed{s(t) = \dfrac{t^3}{16}}$. □

Exercises 4.8.120

Exercises 4.8.120. Consider the figure with solution curves of the given differential equation. Find an equation for the curve through the labeled point.



Solution. Let $y = f(x)$ where $\dfrac{dy}{dx} = \dfrac{df}{dx} = x - 1$, so f is an antiderivative of $x - 1$; that is, $f(x) \in \displaystyle\int x - 1\,dx$.

Exercises 4.8.120

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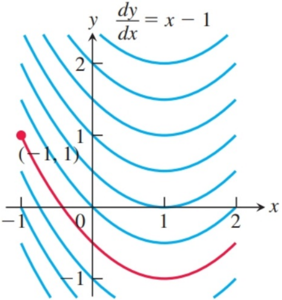

Solution. Let $y = f(x)$ where $\dfrac{dy}{dx} = \dfrac{df}{dx} = x - 1$, so f is an antiderivative of $x - 1$; that is, $f(x) \in \displaystyle\int x - 1\,dx$.

Exercises 4.8.120 (continued)

Solution (continued). Now

$$\int x - 1\,dx = \int x\,dx - \int 1\,dx = \left(\frac{x^2}{2}\right) - x + C = \frac{x^2}{2} - x + C.$$

So $f(x) = \frac{x^2}{2} - x + k$ for some constant k.

Exercises 4.8.120 (continued)

Solution (continued). Now

$$\int x - 1\,dx = \int x\,dx - \int 1\,dx = \left(\frac{x^2}{2}\right) - x + C = \frac{x^2}{2} - x + C.$$

So $f(x) = \dfrac{x^2}{2} - x + k$ for some constant k. The fact that the graph of the desired function f passes through the point $(-1, 1)$ gives us the initial condition $f(-1) = 1$. We use this initial condition to find k. We set $f(-1) = \dfrac{(-1)^2}{2} - (-1) + k = 1$ which requires $3/2 + k = 1$ or $k = -1/2$. So $\boxed{f(x) = \dfrac{x^2}{2} - x - \dfrac{1}{2}}$. □

Exercises 4.8.120 (continued)

Solution (continued). Now

$$\int x - 1\,dx = \int x\,dx - \int 1\,dx = \left(\frac{x^2}{2}\right) - x + C = \frac{x^2}{2} - x + C.$$

So $f(x) = \dfrac{x^2}{2} - x + k$ for some constant k. The fact that the graph of the desired function f passes through the point $(-1, 1)$ gives us the initial condition $f(-1) = 1$. We use this initial condition to find k. We set $f(-1) = \dfrac{(-1)^2}{2} - (-1) + k = 1$ which requires $3/2 + k = 1$ or $k = -1/2$.

So $\boxed{f(x) = \dfrac{x^2}{2} - x - \dfrac{1}{2}}$. $\square$

Exercises 4.8.124

Exercises 4.8.124. Liftoff from Earth.
A rocket lifts off from the surface of the Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later.

Solution. We let $v(t)$ represent the velocity of the rocket in m/sec at time t sec after liftoff. So $v(t)$ is an antiderivative of acceleration $a(t) = 20$ m/sec^2; that is, $v(t) \in \int 20\,dt$. Now $\int 20\,dt = 20t + C$ so $v(t) = 20t + k$ for some constant k. We need an initial value to find k.

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