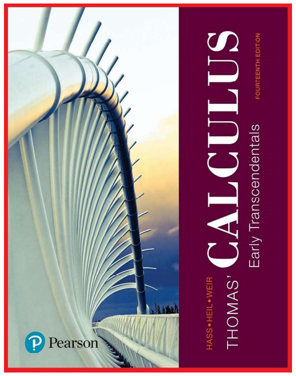
Calculus 1

Chapter 5. Integrals

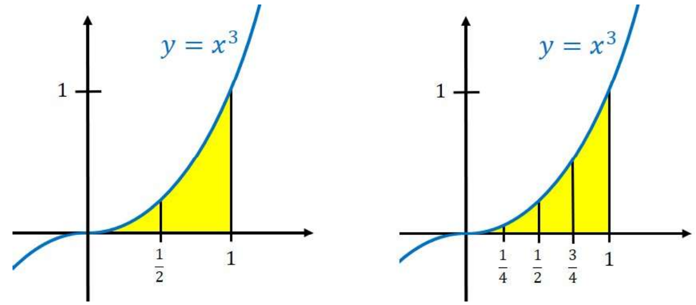
5.1. Area and Estimating with Finite Sums—Examples and Proofs



Exercise 5.1.6

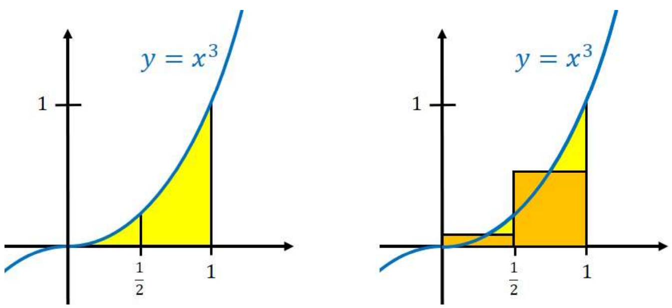
Exercise 5.1.6. Use rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*), estimate the area under the graph of the function $f(x) = x^3$ over the interval $[0, 1]$ using first two and then four rectangles.

Solution. Consider the graph of $y = x^3$ with the interval $[0, 1]$ partitioned into two pieces (left) and four pieces (right).



Exercise 5.1.6 (continued 1)

Solution (continued).



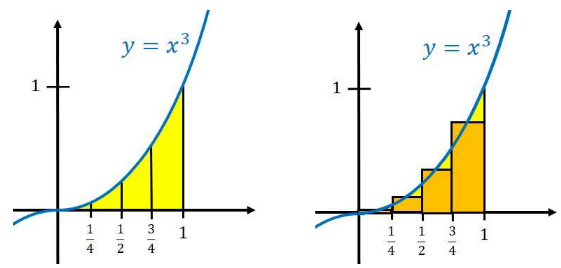
With $n = 2$ subintervals, we have $[0, 1/2]$ and $[1/2, 1]$ which have midpoints $c_1 = 1/4$ and $c_2 = 3/4$. Notice each interval is length $\Delta x = 1/2$. So we have the sum

$$f(c_1)\Delta x + f(c_2)\Delta x = (1/4)^3(1/2) + (3/4)^3(1/2)$$

$$= 1/128 + 27/128 = 28/128 = \boxed{7/32}.$$

Exercise 5.1.6 (continued 2)

Solution (continued).



With $n = 4$ subintervals, we have $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$, and $[3/4, 1]$ which have midpoints $c_1 = 1/8$, $c_2 = 3/8$, $c_3 = 5/8$, and $c_4 = 7/8$, respectively. Notice each interval is length $\Delta x = 1/4$. So we have the sum

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x = (1/8)^3(1/4) + (3/8)^3(1/4)$$

$$+(5/8)^3(1/4) + (7/8)^3(1/4) = (1 + 27 + 125 + 343)/2048 = \boxed{31/128}. \square$$

Exercise 5.1.10

Exercise 5.1.10. Distance Traveled Upstream.
You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with **(a)** left-endpoint values, and **(b)** right-endpoint values.

TIME	0	5	10	15	20	25	30
VELOCITY	1	1.2	1.7	2.0	1.8	1.6	1.4

TIME	35	40	45	50	55	60
VELOCITY	1.2	1.0	1.8	1.5	1.2	0

Here, time is measured in minutes and velocity is measured in centimeters/second.
Solution. We have $n = 12$ subintervals, each of width $\Delta t = 5 \text{ min} = 300 \text{ sec}$.

Exercise 5.1.10 (continued 1)

TIME	0	5	10	15	20	25	30
VELOCITY	1	1.2	1.7	2.0	1.8	1.6	1.4

TIME	35	40	45	50	55	60
VELOCITY	1.2	1.0	1.8	1.5	1.2	0

Solution (continued). (a) Using left-endpoint values (for example, in the subinterval $[0,5]$ we use the velocity at time $t_1 = 0$, in $[5,10]$ we use the velocity at time $t_2 = 5$, etc.) So we have the sum

$$v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_{12})\Delta t = (v(t_1) + v(t_2) + \cdots + v(t_{12}))\Delta t$$

$$= (1 + 1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 + 1.2 + 1.0 + 1.8 + 1.5 + 1.2)(300)$$

$$= (17.4)(300) = \boxed{5220 \text{ cm}}.$$

Exercise 5.1.10 (continued 2)

TIME	0	5	10	15	20	25	30
VELOCITY	1	1.2	1.7	2.0	1.8	1.6	1.4

TIME	35	40	45	50	55	60
VELOCITY	1.2	1.0	1.8	1.5	1.2	0

Solution (continued). (b) Using right-endpoint values (for example, in the subinterval $[0,5]$ we use the velocity at time $t_1 = 5$, in $[5,10]$ we use the velocity at time $t_2 = 10$, etc.) So we have the sum

$$v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_{12})\Delta t = (v(t_1) + v(t_2) + \cdots + v(t_{12}))\Delta t$$

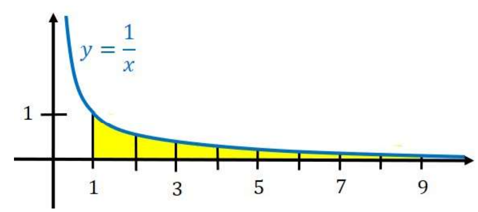
$$= (1.2 + 1.7 + 2.0 + 1.8 + 1.6 + 1.4 + 1.2 + 1.0 + 1.8 + 1.5 + 1.2 + 0)(300)$$

$$= (16.4)(300) = \boxed{4920 \text{ cm}}. \qquad \square$$

Exercise 5.1.16

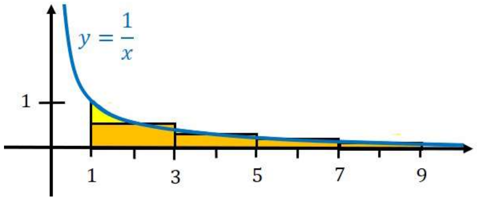
Exercise 5.1.16. Average Value of a Function.
Use a finite sum to estimate the average value of $f(x) = 1/x$ on the interval $[1,9]$ by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

Solution. A sketch of the curve and the area we need to approximate is:



Exercise 5.1.16 (continued)

Solution (continued).



With $n = 4$ subintervals, we have $[1,3]$, $[3,5]$, $[5,7]$, and $[7,9]$ which have midpoints $c_1 = 2$, $c_2 = 4$, $c_3 = 6$, and $c_4 = 8$, respectively. Notice each interval is length $\Delta x = 2$. So we have the sum

$$f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x = (1/(2))(2) + (1/(4))(2)$$

$$+(1/(6))(2) + (1/(8))(2) = 1 + 1/2 + 1/3 + 1/4 = 25/12.$$

So we approximate the average value of f on $[a,b] = [1,9]$ as the approximation of the area divided by $b - a = 9 - 1 = 8$, and we have $\boxed{\text{approximate average value is } 25/96}$. □

Exercise 5.1.20

Exercise 5.1.20. Air Pollution (modified). A power plant generates electricity by burning oil. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere (in tons/day), recorded as follows:

MONTH	JAN	FEB	MAR	APR	MAY	JUN
RATE	0.20	0.25	0.27	0.34	0.45	0.52

MONTH	JUL	AUG	SEP	OCT	NOV	DEC
RATE	0.63	0.70	0.81	0.85	0.89	0.95

(a) Assuming a 30-day month, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate? **(b)** In the worst case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

Proof. We consider the subintervals as the 30-day months. Since the rates increase over the year, an upper estimate would use the right-hand endpoint to estimate tons of pollutants released during the month and a lower estimate would use the left-hand endpoint.

Exercise 5.1.20 (continued 1)

Solution (continued).

MONTH	JAN	FEB	MAR	APR	MAY	JUN
RATE	0.20	0.25	0.27	0.34	0.45	0.52

MONTH	JUL	AUG	SEP	OCT	NOV	DEC
RATE	0.63	0.70	0.81	0.85	0.89	0.95

(a) Using the right-hand endpoints to determine rates $R(t_k)$ corresponds to using the rate at the end of the month. The fact that each month has $\Delta t = 30$ days, gives the sum for the upper estimate as:

$$R(t_1)\Delta t + R(t_2)\Delta t + \cdots + R(t_6)\Delta t = (R(t_1) + R(t_2) + \cdots + R(t_6))\Delta t$$

$$= (0.20 + 0.25 + 0.27 + 0.34 + 0.45 + 0.52)(30) = (2.03)(30) = \boxed{60.9 \text{ tons}}.$$

Exercise 5.1.20 (continued 2)

Solution (continued).

MONTH	JAN	FEB	MAR	APR	MAY	JUN
RATE	0.20	0.25	0.27	0.34	0.45	0.52

MONTH	JUL	AUG	SEP	OCT	NOV	DEC
RATE	0.63	0.70	0.81	0.85	0.89	0.95

Using the left-hand endpoints to determine rates $R(t_k)$ corresponds to using the rate at the beginning of the month (where we use a rate of 0 in January due to the absence of that information). The fact that each month has $\Delta t = 30$ days, gives the sum for the upper estimate as:

$$R(t_1)\Delta t + R(t_2)\Delta t + \cdots + R(t_6)\Delta t = (R(t_1) + R(t_2) + \cdots + R(t_6))\Delta t$$

$$= (0 + 0.20 + 0.25 + 0.27 + 0.34 + 0.45)(30) = (1.51)(30) = \boxed{45.3 \text{ tons}}.$$

Exercise 5.1.20 (continued 3)

Solution (continued).

MONTH	JAN	FEB	MAR	APR	MAY	JUN
RATE	0.20	0.25	0.27	0.34	0.45	0.52

MONTH	JUL	AUG	SEP	OCT	NOV	DEC
RATE	0.63	0.70	0.81	0.85	0.89	0.95

(b) In the worst case, the maximum amount of pollution is released, so the question is: When is the total amount of pollution exceed 125 tons when we use right hand end-points? Since such a sum through June gave 60.9 tons, we now consider such sums for additional months:

$$(0.20 + 0.25 + 0.27 + 0.34 + 0.45 + 0.52 + 0.63)(30)$$
$$= (2.66)(30) = 79.8 \text{ tons (through July)},$$
$$(0.20 + 0.25 + 0.27 + 0.34 + 0.45 + 0.52 + 0.63 + 0.70)(30)$$
$$= (3.36)(30) = 100.8 \text{ tons (through August), and}$$
$$(0.20 + 0.25 + 0.27 + 0.34 + 0.45 + 0.52 + 0.63 + 0.70 + 0.81)(30)$$
$$= (4.17)(30) = 125.1 \text{ tons } \boxed{\text{(through September)}}. \square$$