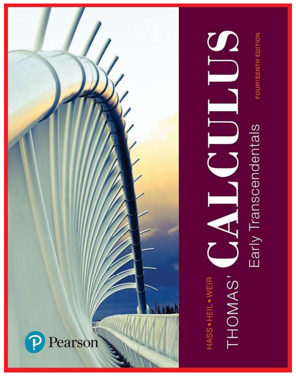
Calculus 1

Chapter 5. Integrals

5.3. The Definite Integral—Examples and Proofs



Exercise 5.3.6

Exercise 5.3.6. Express the limit $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,\Delta x_k$, where P is a partition of $[0,1]$, as a definite integral.

Solution. With $P=\{x_0,x_1,\ldots,x_n\}$ a partition of $[a,b]=[0,1]$, $c_k \in [x_{k-1},x_k]$, $\Delta x_k = x_k - x_{k-1}$, and $f(x)=\sqrt{4-x^2}$ we have that

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,\Delta x_k = \int_a^b f(x)\,dx = \boxed{\int_0^1 \sqrt{4-x^2}\,dx}. \quad \square$$

Example 5.3.1

Example 5.3.1. Show that the function $f(x)=\begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0,1]$.

Solution. If f is Riemann integrable on $[0,1]$ then by the definition of definite integral, $\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$ for any choice of $c_k \in [x_{k-1},x_k]$. Now in any interval $[x_{k-1},x_k]$ there are both rational and irrational numbers. So we can choose each c_k to be rational in which case each $f(c_k)=1$ and $\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k =$ $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (1)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta x_k = \lim_{\|P\|\to 0} 1 = 1$ since the sum of the length of the subintervals is the length of $[0,1]$ (namely 1).

Example 5.3.1 (continued)

Example 5.3.1. Show that the function $f(x)=\begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0,1]$.

Solution (continued). We can also choose each c_k to be irrational in which case each $f(c_k)=0$ and $\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k =$ $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (0)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 0 = \lim_{\|P\|\to 0} 0 = 0$. But we cannot have both $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 f(x)\,dx = 0$, so f is not Riemann integrable over $[0,1]$. $\square$

Theorem 5.2

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
4. *Sum and Difference*: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then
$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$
7. *Domination*: $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Theorem 5.2 (continued 1)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Proof. Let P be a partition of $[a, b]$ and let $\sum_{k=1}^n f(c_k)\,\Delta x_k$ be an associated Riemann sum. Then $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k$ and

$$\begin{aligned}\int_a^b cf(x)\,dx &= \lim_{\|P\|\to 0}\sum_{k=1}^n cf(c_k)\,\Delta x_k \\ &= \lim_{\|P\|\to 0} c\sum_{k=1}^n f(c_k)\,\Delta x_k \text{ since multiplication distributes over addition}\end{aligned}$$

Theorem 5.2 (continued 2)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Proof (continued). ...

$$\begin{aligned}\int_a^b cf(x)\,dx &= \lim_{\|P\|\to 0} c\sum_{k=1}^n f(c_k)\,\Delta x_k \\ &= c\lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k \text{ by the Constant Multiple Rule, Theorem 2.1(3)} \\ &= c\int_a^b f(x)\,dx,\end{aligned}$$

as claimed. □

Theorem 5.2 (continued 3)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

4. *Sum and Difference*: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

Proof (continued). Let P be a partition of $[a, b]$ and let $\sum_{k=1}^n f(c_k)\,\Delta x_k$ and $\sum_{k=1}^n g(c_k)\,\Delta x_k$ be associated Riemann sums. Then by definition
$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k$ and
$\int_a^b g(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n g(c_k)\,\Delta x_k$, so

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k \pm \lim_{\|P\|\to 0}\sum_{k=1}^n g(c_k)\,\Delta x_k$$

Theorem 5.2 (continued 4)

Proof (continued). ...

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k \pm \lim_{\|P\|\to 0} \sum_{k=1}^n g(c_k)\,\Delta x_k$$

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^n f(c_k)\,\Delta x_k \pm \sum_{k=1}^n g(c_k)\,\Delta x_k \right)$$

by the Sum and Difference Rules, Theorem 2.1(1 and 2)

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^n (f(c_k)\,\Delta x_k \pm g(c_k)\,\Delta x_k) \right)$$

by commutivity and addition and subtraction

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^n (f(c_k) \pm g(c_k))\,\Delta x_k \right) \text{ since multiplication}$$

distributes over addition

Theorem 5.2 (continued 5)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a,b]$. Then:

4. *Sum and Difference*:
$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

Proof (continued). ...

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \lim_{\|P\|\to 0} \left(\sum_{k=1}^n (f(c_k) \pm g(c_k))\,\Delta x_k \right) = \int_a^b (f(x) \pm g(x))\,dx,$$

since $\int_a^b (f(x) \pm g(x))\,dx = \lim_{\|P\|\to 0} \left(\sum_{k=1}^n (f(c_k) \pm g(c_k))\,\Delta x_k \right)$, by definition. □

Theorem 5.2 (continued 6)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a,b]$. Then:

6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a,b]$, then
$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

Proof (continued). Let P be a partition of $[a,b]$ and let $\sum_{k=1}^n f(c_k)\,\Delta x_k$ be an associated Riemann sum. Then by definition $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k$. Notice that $\min f \le f(c_k) \le \max f$ for all $c_k \in [a,b]$ and $\sum_{k=1}^n \Delta x_k = (b-a)$.

Theorem 5.2 (continued 7)

Proof (continued). So we have

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k \ge \lim_{\|P\|\to 0} \sum_{k=1}^n \min f\,\Delta x_k$$

$$= \lim_{\|P\|\to 0} \min f \sum_{k=1}^n \Delta x_k \text{ since multiplication}$$

distributes over addition

$$= \min f \lim_{\|P\|\to 0} \sum_{k=1}^n \Delta x_k \text{ by the Constant Multiple Rule,}$$

Theorem 2.1(3)

$$= \min f \lim_{\|P\|\to 0} (b-a) \text{ since } \sum_{k=1}^n \Delta x_k = b-a$$

$$= \min f \cdot (b-a),$$

as claimed.

Theorem 5.2 (continued 8)

Proof (continued). Similarly,

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k \le \lim_{\|P\|\to 0}\sum_{k=1}^n \max f\,\Delta x_k \\ &= \lim_{\|P\|\to 0} \max f \sum_{k=1}^n \Delta x_k \text{ since multiplication} \\ &\quad\text{distributes over addition} \\ &= \max f \lim_{\|P\|\to 0}\sum_{k=1}^n \Delta x_k \text{ by the Constant Multiple Rule,} \\ &\quad\text{Theorem 2.1(3)} \\ &= \max f \lim_{\|P\|\to 0}(b-a) \text{ since } \sum_{k=1}^n \Delta x_k = b-a \\ &= \max f\cdot(b-a),\end{aligned}$$

as claimed. □

Theorem 5.2 (continued 9)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a,b]$. Then:

7. *Domination*: $f(x)\ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Proof (continued). Let P be a partition of $[a,b]$ and let $\sum_{k=1}^n f(c_k)\,\Delta x_k$ and $\sum_{k=1}^n g(c_k)\,\Delta x_k$ be associated Riemann sums. Then by definition $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k$ and $\int_a^b g(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n g(c_k)\,\Delta x_k$. Since $f(x)\ge g(x)$ on $[a,b]$ then $f(c_k)\ge g(c_k)$ for all $c_k\in[a,b]$, and so

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^n f(c_k)\,\Delta x_k \ge \lim_{\|P\|\to 0}\sum_{k=1}^n g(c_k)\,\Delta x_k = \int_a^b g(x)\,dx. \ \square$$

Exercise 5.3.10

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(a)** $\int_1^9 -2f(x)\,dx$, **(b)** $\int_7^9 (f(x)-h(x))\,dx$, **(c)** $\int_7^9 (2f(x)-3h(x))\,dx$, **(d)** $\int_9^1 f(x)\,dx$, **(e)** $\int_1^7 f(x)\,dx$, and **(f)** $\int_9^7 (h(x)-f(x))\,dx$.

Solution. (a) We have

$$\begin{aligned}\int_1^9 -2f(x)\,dx &= -2\int_1^9 f(x)\,dx \text{ by Constant Multiple Rule, Thm 5.2(3)} \\ &= -2(-1) = 2 \text{ since } \int_1^9 f(x)\,dx = -1. \quad \square\end{aligned}$$

Exercise 5.3.10 (continued 1)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(b)** $\int_7^9 (f(x)-h(x))\,dx$.

Solution (continued). (b) We have

$$\begin{aligned}\int_7^9 (f(x)-h(x))\,dx &= \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx \\ &\quad\text{by the Difference Rule, Theorem 5.2(4)} \\ &= (5)-(4) = 1 \text{ since } \int_7^9 f(x)\,dx = 5 \\ &\quad\text{and } \int_7^9 h(x)\,dx = 4. \quad \square\end{aligned}$$

Exercise 5.3.10 (continued 2)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(c)** $\int_7^9 (2f(x) - 3h(x))\,dx$.

Solution (continued). (c) We have $\int_7^9 (2f(x) - 3h(x))\,dx =$

$$\begin{aligned} &= \int_7^9 2f(x)\,dx + \int_7^9 -3h(x)\,dx \text{ by the Sum Rule, Theorem 5.2(4)} \\ &= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx \text{ by Constant Mult. Theorem 5.2(3)} \\ &= 2(5) - 3(4) = -2 \text{ since } \int_7^9 f(x)\,dx = 5 \text{ and } \int_7^9 h(x)\,dx = 4. \quad \square \end{aligned}$$

Exercise 5.3.10 (continued 3)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(d)** $\int_9^1 f(x)\,dx$.

Solution (continued). (d) We have

$$\begin{aligned} \int_9^1 f(x)\,dx &= -\int_1^9 f(x)\,dx \text{ by the Order of Integration, Theorem 5.2(1)} \\ &= -(-1) = 1 \text{ since } \int_1^9 f(x)\,dx = -1. \quad \square \end{aligned}$$

Exercise 5.3.10 (continued 4)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(e)** $\int_1^7 f(x)\,dx$.

Solution (continued). (e) By Additivity (Theorem 5.2(5)) we have $\int_1^7 f(x)\,dx + \int_7^9 f(x)\,dx = \int_1^9 f(x)\,dx$, then $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$. So $\int_1^7 f(x)\,dx = (-1) - (5) = \boxed{-6}$, since $\int_1^9 f(x)\,dx = -1$ and $\int_7^9 f(x)\,dx = 5$. $\square$

Exercise 5.3.10 (continued 5)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(f)** $\int_9^7 (h(x) - f(x))\,dx$.

Solution (continued). (f) We have $\int_9^7 (h(x) - f(x))\,dx =$

$$\begin{aligned} &= \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx \text{ by the Difference Rule, Theorem 5.2(4)} \\ &= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx \text{ by Order of Integration, Theorem 5.2(1)} \\ &= -(4) + (5) = \boxed{1} \text{ since } \int_7^9 f(x)\,dx = 5 \text{ and } \int_7^9 h(x)\,dx = 4. \quad \square \end{aligned}$$

Exercise 5.3.63

Exercise 5.3.63. Let c be a constant. Prove that $\int_a^b c\,dx = c(b-a)$.

Proof. Let $f(x) = c$. Then f is continuous on $[a,b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a,b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ (see Note 5.3.A). Now $\|P\| = \Delta x = (b-a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$. So the value of the Riemann integral is given by

$$\begin{aligned}
\int_a^b c\,dx &= \lim_{n\to\infty} \sum_{k=1}^n f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^n c\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left(nc\frac{b-a}{n}\right) \text{ by Theorem 5.2.A(4)} \\
&= \lim_{n\to\infty} c(b-a) = c(b-a). \quad \square
\end{aligned}$$

Example 5.3.A

Example 5.3.A. Use a regular partition of $[a,b]$ with $c_k = x_k$ to prove that for $a < b$: $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$.

Proof. Let $f(x) = x$. Then f is continuous on $[a,b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a,b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b-a)/n$ (see Note 5.3.A). Now $\|P\| = \Delta x = (b-a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$. So the value of the Riemann integral is given by

$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^n c_k\left(\frac{b-a}{n}\right)$$

Example 5.3.A (continued 1)

Proof (continued).

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty} \sum_{k=1}^n c_k\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^n \left(a + k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left(\sum_{k=1}^n a + \frac{b-a}{n}\sum_{k=1}^n k\right) \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left((na) + \frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right) \\
&\quad \text{since } \sum_{k=1}^n k = \frac{n(n+1)}{2} \\
&= \lim_{n\to\infty} \left((b-a)a + \left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)}{2}\right)\right)
\end{aligned}$$

Example 5.3.A (continued 2)

Proof (continued).

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty} \left((b-a)a + \left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)}{2}\right)\right) \\
&= (b-a)a + (b-a)^2 \lim_{n\to\infty} \frac{n(n+1)}{2n^2} \\
&= (b-a)a + (b-a)^2 \lim_{n\to\infty} \frac{n^2+n}{2n^2}\left(\frac{1/n^2}{1/n^2}\right) \\
&= (b-a)a + (b-a)^2 \lim_{n\to\infty} \frac{1+1/n}{2} \\
&= (b-a)a + (b-a)^2 \frac{1 + \lim_{n\to\infty} 1/n}{2} \\
&= (b-a)a + (b-a)^2 \frac{1+(0)}{2} = ab - a^2 + \frac{b^2 - 2ab + a^2}{2} \\
&= \frac{b^2}{2} - \frac{a^2}{2}. \quad \square
\end{aligned}$$

Exercise 5.3.65

Exercise 5.3.65. Use a regular partition of $[a,b]$ with $c_k = x_k$ to prove that for $a < b$: $\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

Proof. Let $f(x) = x^2$. Then f is continuous on $[a,b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a,b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b-a)/n$ (see Note 5.3.A). Now $\|P\| = \Delta x = (b-a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$. So the value of the Riemann integral is given by

$$\int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^n f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^n c_k^2\left(\frac{b-a}{n}\right)$$

Exercise 5.3.65 (continued 1)

Proof (continued).

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty} \sum_{k=1}^n c_k^2\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^n \left(a + k\frac{b-a}{n}\right)^2\left(\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{k=1}^n\left(a^2 + 2ak\frac{b-a}{n} + k^2\left(\frac{b-a}{n}\right)^2\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2) + 2a\frac{b-a}{n}\sum_{k=1}^n k + \left(\frac{b-a}{n}\right)^2\sum_{k=1}^n k^2\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2) + 2a\frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right.\\
&\quad \left. + \left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)(2n+1)}{6}\right)\right)
\end{aligned}$$

since $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$

Exercise 5.3.65 (continued 2)

Proof (continued).

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2) + 2a\frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right.\\
&\quad \left. + \left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\
&= \lim_{n\to\infty}(b-a)\left(a^2 + 2a\frac{b-a}{n^2}\left(\frac{n(n+1)}{2}\right)\right.\\
&\quad \left. + \frac{(b-a)^2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\
&= \lim_{n\to\infty}(b-a)\left(a^2 + 2a(b-a)\left(\frac{(n^2+n)/n^2}{2}\right)\right.\\
&\quad \left. +(b-a)^2\left(\frac{(2n^3+3n^2+n)/n^3}{6}\right)\right)
\end{aligned}$$

Exercise 5.3.65 (continued 3)

Proof (continued).

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty}(b-a)\left(a^2 + 2a(b-a)\left(\frac{(n^2+n)/n^2}{2}\right)\right.\\
&\quad \left. +(b-a)^2\left(\frac{(2n^3+3n^2+n)/n^3}{6}\right)\right)\\
&= \lim_{n\to\infty}(b-a)\left(a^2 + 2a(b-a)\left(\frac{1+1/n}{2}\right)\right.\\
&\quad \left. +(b-a)^2\left(\frac{2+3/n+1/n^2}{6}\right)\right)\\
&= (b-a)\left(a^2 + 2a(b-a)\left(\frac{1+\lim_{n\to\infty}1/n}{2}\right)\right.\\
&\quad \left. +(b-a)^2\left(\frac{2+3\lim_{n\to\infty}(1/n)+(\lim_{n\to\infty}1/n)^2}{6}\right)\right)
\end{aligned}$$

Exercise 5.3.65 (continued 4)

Proof (continued).

$$\begin{aligned}\int_a^b x^2\,dx &= (b-a)\left(a^2+2a(b-a)\left(\frac{1+(0)}{2}\right)\right.\\ &\quad \left.+(b-a)^2\left(\frac{2+3(0)+(0)^2}{6}\right)\right)\\ &= (b-a)\left(a^2+a(b-a)+(b-a)^2(1/3)\right)\\ &= (b-a)(a^2+ab-a^2+b^2/3-2ab/3+a^2/3)\\ &= (b-a)(ab/3+b^2/3+a^2/3)\\ &= (ab^2+b^3+a^2b-a^2b-ab^2-a^3)/3\\ &= \frac{b^3}{3}-\frac{a^3}{3}. \quad \square\end{aligned}$$

Exercise 5.3.36

Exercise 5.3.36. Use Equation (4) (see Exercise 5.3.65) to evaluate the integral $\int_0^{\pi/2} \theta^2\,d\theta$.

Solution. The integrand is $f(\theta)=\theta^2$, the lower bound of the integral is $a=0$, and the upper bound of the integral is $b=\pi/2$. So by Equation (4) (Exercise 5.3.65),

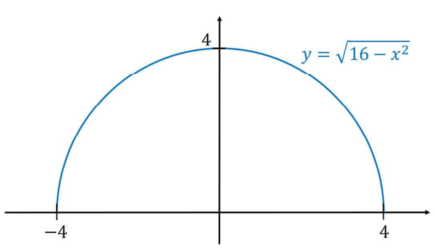
$$\int_0^{\pi/2} \theta^2\,d\theta = \frac{b^3}{3}-\frac{a^3}{3} = \frac{(\pi/2)^3}{3}-\frac{0^3}{3} = \boxed{\frac{\pi^3}{24}}.$$

$\square$

Exercise 5.3.18

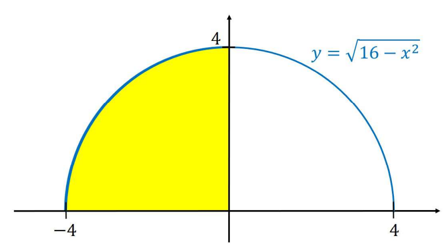
Exercise 5.3.18. Graph the integrand and use known area formulas to evaluate the integral: $\int_{-4}^{0} \sqrt{16-x^2}\,dx$.

Solution. Notice that with $y=\sqrt{16-x^2}$, we have $y^2=(\sqrt{16-x^2})^2=16-x^2$ and $y\ge 0$. So $x^2+y^2=16$ and $y\ge 0$. So the graph of $y=\sqrt{16-x^2}$ is the top half (since $y\ge 0$) of a circle of radius $r=4$ and center $(0,0)$:



Exercise 5.3.18 (continued)

Solution.



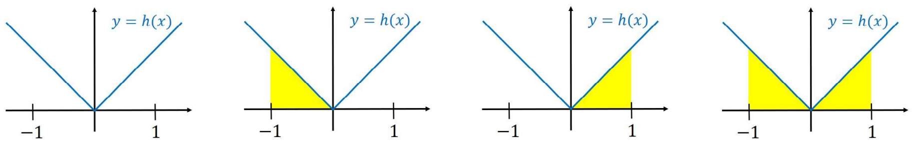
Since $y=f(x)=\sqrt{16-x^2}$ is non-negative, then (by definition) the definite integral $\int_{-4}^{0}\sqrt{16-x^2}\,dx$ is the area under the curve $y=\sqrt{16-x^2}$ (and above the x-axis) from $a=-4$ to $b=0$. That is, the integral is $1/4$ of the area of a circle of radius $r=4$. Therefore,

$$\int_{-4}^{0}\sqrt{16-x^2}\,dx = \left.\frac{\pi(r)^2}{4}\right|_{r=4} = \frac{\pi(4)^2}{4} = \boxed{4\pi}. \quad \square$$

Exercise 5.3.62

Exercise 5.3.62. Graph the function $h(x) = |x|$ and find the average value over the intervals **(a)** $[-1, 0]$, **(b)** $[0, 1]$, and **(c)** $[-1, 1]$.

Solution. We consider the graph and relevant areas:



By definition $\text{av}(h) = \frac{1}{b-a}\int_a^b h(x)\,dx$, so calculating the integrals using areas we have

(a) The average of h over $[-1, 0]$ is $\frac{1}{(0)-(-1)}\int_{-1}^0 h(x)\,dx = \boxed{1/2}$.

(b) The average of h over $[0, 1]$ is $\frac{1}{(1)-(0)}\int_0^1 h(x)\,dx = \boxed{1/2}$.

(c) The average of h over $[-1, 1]$ is

$\frac{1}{(1)-(-1)}\int_{-1}^1 h(x)\,dx = 1/2(1) = \boxed{1/2}$. □

Exercise 5.3.76

Exercise 5.3.76. Show that the value of $\displaystyle\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Solution. Let $f(x) = \sqrt{x+8} = (x+8)^{1/2}$. Then $f'(x) = \dfrac{1}{2}(x+8)^{-1/2} = \dfrac{1}{2\sqrt{x+8}}$ and so the only critical point of f is $x = -8$. So continuous function f has no critical points in $[0, 1]$ and hence by the technique of Section 4.1, "Extreme Values of Functions on Closed Intervals," the extremes of f on $[0, 1]$ occur at the endpoints. Since $f(0) = \sqrt{(0)+8} = \sqrt{8} = 2\sqrt{2}$ and $f(1) = \sqrt{(1)+8} = \sqrt{9} = 3$, then the minimum of f on $[a, b] = [0, 1]$ is $\min f = 2\sqrt{2}$ and the maximum is $\min f = 3$.

Exercise 5.3.76 (continued)

Exercise 5.3.76. Show that the value of $\displaystyle\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Solution (continued). By Theorem 5.2(6), the Max-Min Inequality, we have

$$\min f \cdot (b-a) = (2\sqrt{2})((1)-(0)) \leq \int_a^b f(x)\,dx$$

$$= \int_0^1 \sqrt{x+8}\,dx \leq \max f \cdot (b-a) = (3)((1)-(0),$$

of $2\sqrt{2} \leq \displaystyle\int_0^1 \sqrt{x+8}\,dx \leq 3$, as claimed. □

Exercise 5.3.88

Exercise 5.3.88. If you average 30 miles/hour on a 150 mile trip and then return over the same 150 miles at the rate of 50 miles/hour, what is your average speed for the trip? Give reasons for your answer.

Solution. We define function $f(t)$ as your speed as a function of time t, where t is measured in hours and f is measured in miles/hour. So we have f defined piecewise as $f(t) = 30$ miles/hour for t between 0 hours and 5 hours (since it takes 5 hours to travel 150 miles at 30 miles/hour) and $f(t) = 50$ miles/hour for t between 5 hours and 8 hours (since it takes 3 hours to travel 150 miles at 50 miles/hour): $f(t) = \begin{cases} 30, & 0 \leq t < 5 \\ 50, & 5 \leq t \leq 8 \end{cases}$

Exercise 5.3.88 (continued)

Solution (continued). $\ldots f(t) = \begin{cases} 30, & 0 \le t < 5 \\ 50, & 5 \le t \le 8 \end{cases}$ So, by definition, the average speed (i.e., the average of f over $[0,8]$) is

$$\begin{aligned} \mathrm{av}(f) &= \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{(8)-(0)}\int_0^8 f(t)\,dt \\ &= \frac{1}{8}\left(\int_0^5 f(t)\,dt + \int_5^8 f(t)\,dt\right) \text{ by Theorem 5.2(5), Additivity} \\ &= \frac{1}{8}\left(\int_0^5 30\,dt + \int_5^8 50\,dt\right) \\ &= \frac{1}{8}((30)((5)-(0)) + (50)((8)-(5))) \text{ by Exercise 5.3.63} \\ &= \frac{1}{8}(150+150) = \frac{300}{8} = \boxed{\frac{75}{2} \text{ miles/hour.}} \end{aligned}$$

□