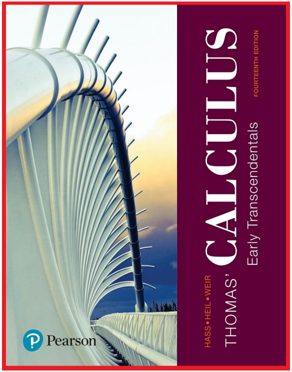
Calculus 1

Chapter 5. Integrals

5.3. The Definite Integral—Examples and Proofs



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Exercise 5.3.6. Express the limit $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,\Delta x_k$, where P is a partition of $[0,1]$, as a definite integral.

Solution. With $P = \{x_0, x_1, \ldots, x_n\}$ a partition of $[a,b] = [0,1]$, $c_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, and $f(x) = \sqrt{4-x^2}$ we have that

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4-c_k^2}\,\Delta x_k = \int_a^b f(x)\,dx = \boxed{\int_0^1 \sqrt{4-x^2}\,dx}. \qquad \square$$

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Example 5.3.1

Example 5.3.1. Show that the function $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0, 1]$.

Solution. If f is Riemann integrable on $[0, 1]$ then by the definition of definite integral, $\int_0^1 f(x)\, dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\, \Delta x_k$ for any choice of $c_k \in [x_{k-1}, x_k]$. Now in any interval $[x_{k-1}, x_k]$ there are both rational and irrational numbers.

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$\lim_{\|P\|\to 0} \sum_{k=1}^{n} (1)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta x_k = \lim_{\|P\|\to 0} 1 = 1$ since the sum of the length of the subintervals is the length of $[0, 1]$ (namely 1).

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Example 5.3.1 (continued)

Example 5.3.1. Show that the function $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0, 1]$.

Solution (continued). We can also choose each c_k to be irrational in which case each $f(c_k) = 0$ and $\displaystyle\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} (0)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^{n} 0 = \lim_{\|P\|\to 0} 0 = 0$. But we cannot have both $\displaystyle\int_0^1 f(x)\,dx = 1$ and $\displaystyle\int_0^1 f(x)\,dx = 0$, so f is not Riemann integrable over $[0, 1]$. □

Example 5.3.1 (continued)

Example 5.3.1. Show that the function $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is not Riemann integrable over the interval $[0,1]$.

Solution (continued). We can also choose each c_k to be irrational in which case each $f(c_k) = 0$ and $\int_0^1 f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k =$ $\lim_{\|P\|\to 0} \sum_{k=1}^n (0)\,\Delta x_k = \lim_{\|P\|\to 0} \sum_{k=1}^n 0 = \lim_{\|P\|\to 0} 0 = 0$. But we cannot have both $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 f(x)\,dx = 0$, so f is not Riemann integrable over $[0,1]$. $\square$

Theorem 5.2

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
4. *Sum and Difference*:
$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination*: $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx.$

Theorem 5.2 (continued 1)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Proof. Let P be a partition of $[a, b]$ and let $\displaystyle\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ be an associated Riemann sum. Then $\displaystyle\int_a^b f(x)\,dx = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ and

$$\begin{aligned}\int_a^b cf(x)\,dx &= \lim_{\|P\|\to 0}\sum_{k=1}^{n} cf(c_k)\,\Delta x_k \\ &= \lim_{\|P\|\to 0} c\sum_{k=1}^{n} f(c_k)\,\Delta x_k \text{ since multiplication} \\ &\quad \text{distributes over addition}\end{aligned}$$

Theorem 5.2 (continued 2)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

3. *Constant Multiple*: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

Proof (continued). ...

$$\begin{aligned}\int_a^b cf(x)\,dx &= \lim_{\|P\|\to 0} c\sum_{k=1}^{n} f(c_k)\,\Delta x_k \\ &= c\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\,\Delta x_k \text{ by the Constant Multiple Rule,} \\ &\quad \text{Theorem 2.1(3)} \\ &= c\int_a^b f(x)\,dx,\end{aligned}$$

as claimed. □

Theorem 5.2 (continued 3)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

4. *Sum and Difference*:
$$\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Proof (continued). Let P be a partition of $[a, b]$ and let $\displaystyle\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ and $\displaystyle\sum_{k=1}^{n} g(c_k)\,\Delta x_k$ be associated Riemann sums. Then by definition $\int_a^b f(x)\, dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k$ and $\int_a^b g(x)\, dx = \lim_{\|P\|\to 0} \sum_{k=1}^n g(c_k)\,\Delta x_k$, so

$$\int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \pm \lim_{\|P\|\to 0} \sum_{k=1}^{n} g(c_k)\,\Delta x_k$$

Theorem 5.2 (continued 4)

Proof (continued). ...

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \pm \lim_{\|P\|\to 0} \sum_{k=1}^{n} g(c_k)\,\Delta x_k$$

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^{n} f(c_k)\,\Delta x_k \pm \sum_{k=1}^{n} g(c_k)\,\Delta x_k \right)$$

by the Sum and Difference Rules, Theorem 2.1(1 and 2)

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^{n} (f(c_k)\,\Delta x_k \pm g(c_k)\,\Delta x_k) \right)$$

by commutivity and addition and subtraction

$$= \lim_{\|P\|\to 0} \left(\sum_{k=1}^{n} (f(c_k) \pm g(c_k))\ \Delta x_k \right)$$ since multiplication

distributes over addition

Theorem 5.2 (continued 5)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

4. *Sum and Difference*:
$$\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Proof (continued). ...

$$\begin{aligned} \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx &= \lim_{\|P\| \to 0} \left(\sum_{k=1}^{n} (f(c_k) \pm g(c_k))\ \Delta x_k \right) \\ &= \int_a^b (f(x) \pm g(x))\, dx, \end{aligned}$$

since $\int_a^b (f(x) \pm g(x))\, dx = \lim_{\|P\| \to 0} \left(\sum_{k=1}^{n} (f(c_k) \pm g(c_k))\ \Delta x_k \right)$, by definition. □

Theorem 5.2 (continued 6)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

6. *Max-Min Inequality*: If max f and min f are the maximum and minimum values of f on $[a, b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b-a).$$

Proof (continued). Let P be a partition of $[a, b]$ and let $\displaystyle\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ be an associated Riemann sum. Then by definition $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k$. Notice that

$\min f \leq f(c_k) \leq \max f$ for all $c_k \in [a, b]$ and $\displaystyle\sum_{k=1}^{n} \Delta x_k = (b-a)$.

Theorem 5.2 (continued 7)

Proof (continued). So we have

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k \geq \lim_{\|P\|\to 0} \sum_{k=1}^n \min f\, \Delta x_k \\
&= \lim_{\|P\|\to 0} \min f \sum_{k=1}^n \Delta x_k \text{ since multiplication} \\
&\quad \text{distributes over addition} \\
&= \min f \lim_{\|P\|\to 0} \sum_{k=1}^n \Delta x_k \text{ by the Constant Multiple Rule,} \\
&\quad \text{Theorem 2.1(3)} \\
&= \min f \lim_{\|P\|\to 0} (b-a) \text{ since } \sum_{k=1}^n \Delta x_k = b-a \\
&= \min f \cdot (b-a),
\end{aligned}$$

as claimed.

Theorem 5.2 (continued 8)

Proof (continued). Similarly,

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \le \lim_{\|P\|\to 0} \sum_{k=1}^{n} \max f\,\Delta x_k \\
&= \lim_{\|P\|\to 0} \max f \sum_{k=1}^{n} \Delta x_k \text{ since multiplication} \\
&\quad \text{distributes over addition} \\
&= \max f \lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta x_k \text{ by the Constant Multiple Rule,} \\
&\quad \text{Theorem 2.1(3)} \\
&= \max f \lim_{\|P\|\to 0} (b-a) \text{ since } \sum_{k=1}^{n} \Delta x_k = b-a \\
&= \max f \cdot (b-a),
\end{aligned}$$

as claimed. □

Theorem 5.2 (continued 9)

Theorem 5.2. Rules Satisfied by Definite Integrals. Suppose f and g are integrable over the interval $[a, b]$. Then:

7. *Domination*: $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Proof (continued). Let P be a partition of $[a, b]$ and let $\sum_{k=1}^{n} f(c_k)\,\Delta x_k$ and $\sum_{k=1}^{n} g(c_k)\,\Delta x_k$ be associated Riemann sums. Then by definition $\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n f(c_k)\,\Delta x_k$ and $\int_a^b g(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^n g(c_k)\,\Delta x_k$. Since $f(x) \geq g(x)$ on $[a, b]$ then $f(c_k) \geq g(c_k)$ for all $c_k \in [a, b]$, and so

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k \geq \lim_{\|P\|\to 0} \sum_{k=1}^{n} g(c_k)\,\Delta x_k = \int_a^b g(x)\,dx. \square$$

Exercise 5.3.10

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(a)** $\int_1^9 -2f(x)\,dx$, **(b)** $\int_7^9 (f(x) - h(x))\,dx$, **(c)** $\int_7^9 (2f(x) - 3h(x))\,dx$, **(d)** $\int_9^1 f(x)\,dx$, **(e)** $\int_1^7 f(x)\,dx$, and **(f)** $\int_9^7 (h(x) - f(x))\,dx$.

Solution. **(a)** We have

$$\begin{aligned}\int_1^9 -2f(x)\,dx &= -2\int_1^9 f(x)\,dx \text{ by Constant Multiple Rule, Thm 5.2(3)} \\ &= -2(-1) = 2 \text{ since } \int_1^9 f(x)\,dx = -1. \quad \square\end{aligned}$$

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Exercise 5.3.10 (continued 1)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(b)** $\int_7^9 (f(x) - h(x))\,dx$.

Solution (continued). (b) We have

$$\begin{aligned} \int_7^9 (f(x) - h(x))\,dx &= \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx \\ &\quad \text{by the Difference Rule, Theorem 5.2(4)} \\ &= (5) - (4) = 1 \text{ since } \int_7^9 f(x)\,dx = 5 \\ &\quad \text{and } \int_7^9 h(x)\,dx = 4. \quad \square \end{aligned}$$

Exercise 5.3.10 (continued 2)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(c)** $\int_7^9 (2f(x) - 3h(x))\,dx$.

Solution (continued). **(c)** We have $\int_7^9 (2f(x) - 3h(x))\,dx =$

$$\begin{aligned} &= \int_7^9 2f(x)\,dx + \int_7^9 -3h(x)\,dx \text{ by the Sum Rule, Theorem 5.2(4)} \\ &= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx \text{ by Constant Mult. Theorem 5.2(3)} \\ &= 2(5) - 3(4) = -2 \text{ since } \int_7^9 f(x)\,dx = 5 \text{ and } \int_7^9 h(x)\,dx = 4. \quad \square \end{aligned}$$

Exercise 5.3.10 (continued 3)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(d)** $\int_9^1 f(x)\,dx$.

Solution (continued). (d) We have

$$\begin{aligned} \int_9^1 f(x)\,dx &= -\int_1^9 f(x)\,dx \text{ by the Order of Integration, Theorem 5.2(1)} \\ &= -(-1) = 1 \text{ since } \int_1^9 f(x)\,dx = -1. \quad \square \end{aligned}$$

Exercise 5.3.10 (continued 4)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(e)** $\int_1^7 f(x)\,dx$.

Solution (continued). (e) By Additivity (Theorem 5.2(5)) we have $\int_1^7 f(x)\,dx + \int_7^9 f(x)\,dx = \int_1^9 f(x)\,dx$, then $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx$. So $\int_1^7 f(x)\,dx = (-1) - (5) = \boxed{-6}$, since $\int_1^9 f(x)\,dx = -1$ and $\int_7^9 f(x)\,dx = 5$. $\square$

Exercise 5.3.10 (continued 5)

Exercise 5.3.10. Suppose that f is h are integrable and that $\int_1^9 f(x)\,dx = -1$, $\int_7^9 f(x)\,dx = 5$, and $\int_7^9 h(x)\,dx = 4$. Use the rules in Theorem 5.2 to find **(f)** $\int_9^7 (h(x) - f(x))\,dx$.

Solution (continued). (f) We have $\int_9^7 (h(x) - f(x))\,dx =$

$$= \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx \text{ by the Difference Rule, Theorem 5.2(4)}$$

$$= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx \text{ by Order of Integration, Theorem 5.2(1)}$$

$$= -(4) + (5) = \boxed{1} \text{ since } \int_7^9 f(x)\,dx = 5 \text{ and } \int_7^9 h(x)\,dx = 4. \quad \square$$

Exercise 5.3.63

Exercise 5.3.63. Let c be a constant. Prove that $\displaystyle\int_a^b c\,dx = c(b-a)$.

Proof. Let $f(x) = c$. Then f is continuous on $[a, b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a, b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ (see Note 5.3.A).

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$$\begin{aligned}\int_a^b c\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} c\left(\frac{b-a}{n}\right)\\ &= \lim_{n\to\infty}\left(nc\frac{b-a}{n}\right) \text{ by Theorem 5.2.A(4)}\\ &= \lim_{n\to\infty} c(b-a) = c(b-a). \quad \square\end{aligned}$$

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Proof. Let $f(x) = c$. Then f is continuous on $[a,b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a,b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ (see Note 5.3.A). Now $\|P\| = \Delta x = (b-a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$. So the value of the Riemann integral is given by

$$\begin{aligned}
\int_a^b c\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} c\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left(nc\frac{b-a}{n}\right) \text{ by Theorem 5.2.A(4)} \\
&= \lim_{n\to\infty} c(b-a) = c(b-a). \quad \square
\end{aligned}$$

Example 5.3.A

Example 5.3.A. Use a regular partition of $[a,b]$ with $c_k = x_k$ to prove that for $a < b$: $\displaystyle\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$.

Proof. Let $f(x) = x$. Then f is continuous on $[a,b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a,b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b-a)/n$, $x_k = a + k(b-a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b-a)/n$ (see Note 5.3.A).

Example 5.3.A

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Proof. Let $f(x) = x$. Then f is continuous on $[a, b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a, b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b - a)/n$, $x_k = a + k(b - a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b - a)/n$ (see Note 5.3.A). Now $\|P\| = \Delta x = (b - a)/n$, so when $n \to \infty$ we have $\|P\| \to 0$. So the value of the Riemann integral is given by

$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k\left(\frac{b-a}{n}\right)$$

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Example 5.3.A (continued 1)

Proof (continued).

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} c_k\left(\frac{b-a}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(a+k\frac{b-a}{n}\right)\left(\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left(\sum_{k=1}^{n} a + \frac{b-a}{n}\sum_{k=1}^{n} k\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na)+\frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right)\\
&\quad \text{since } \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\\
&= \lim_{n\to\infty}\left((b-a)a+\left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)}{2}\right)\right)
\end{aligned}$$

Example 5.3.A (continued 2)

Proof (continued).

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{n\to\infty}\left((b-a)a+\left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)}{2}\right)\right)\\
&= (b-a)a+(b-a)^2\lim_{n\to\infty}\frac{n(n+1)}{2n^2}\\
&= (b-a)a+(b-a)^2\lim_{n\to\infty}\frac{n^2+n}{2n^2}\left(\frac{1/n^2}{1/n^2}\right)\\
&= (b-a)a+(b-a)^2\lim_{n\to\infty}\frac{1+1/n}{2}\\
&= (b-a)a+(b-a)^2\frac{1+\lim_{n\to\infty}1/n}{2}\\
&= (b-a)a+(b-a)^2\frac{1+(0)}{2} = ab-a^2+\frac{b^2-2ab+a^2}{2}\\
&= \frac{b^2}{2}-\frac{a^2}{2}. \qquad \square
\end{aligned}$$

Exercise 5.3.65

Exercise 5.3.65. Use a regular partition of $[a, b]$ with $c_k = x_k$ to prove that for $a < b$: $\displaystyle\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$.

Proof. Let $f(x) = x^2$. Then f is continuous on $[a, b]$ so, by "Integrability of Continuous Functions" (Theorem 5.1), f is integrable on $[a, b]$. Therefore, we can consider any sequence of partitions which have a norm approaching 0. So we consider an equal width partition $P = \{x_0, x_1, \ldots, x_n\}$ for which $\Delta x_k = \Delta x = (b - a)/n$, $x_k = a + k(b - a)/n$, and $c_k \in [x_{k-1}, x_k]$ satisfies $c_k = x_k = a + k(b - a)/n$ (see Note 5.3.A).

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$$\int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k^2\left(\frac{b-a}{n}\right)$$

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$$\int_a^b x^2\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} c_k^2 \left(\frac{b-a}{n}\right)$$

Exercise 5.3.65 (continued 1)

Proof (continued).

$$\begin{aligned}\int_a^b x^2\,dx &= \lim_{n\to\infty}\sum_{k=1}^{n} c_k^2\left(\frac{b-a}{n}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(a+k\frac{b-a}{n}\right)^2\left(\frac{b-a}{n}\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\sum_{k=1}^{n}\left(a^2+2ak\frac{b-a}{n}+k^2\left(\frac{b-a}{n}\right)^2\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2)+2a\frac{b-a}{n}\sum_{k=1}^{n}k+\left(\frac{b-a}{n}\right)^2\sum_{k=1}^{n}k^2\right)\\
&= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2)+2a\frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right.\\
&\qquad\left.+\left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\end{aligned}$$

since $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Exercise 5.3.65 (continued 2)

Proof (continued).

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{n\to\infty}\left(\frac{b-a}{n}\right)\left((na^2)+2a\frac{b-a}{n}\left(\frac{n(n+1)}{2}\right)\right.\\
&\qquad \left.+\left(\frac{b-a}{n}\right)^2\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\
&= \lim_{n\to\infty}(b-a)\left(a^2+2a\frac{b-a}{n^2}\left(\frac{n(n+1)}{2}\right)\right.\\
&\qquad \left.+\frac{(b-a)^2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right)\\
&= \lim_{n\to\infty}(b-a)\left(a^2+2a(b-a)\left(\frac{(n^2+n)/n^2}{2}\right)\right.\\
&\qquad \left.+(b-a)^2\left(\frac{(2n^3+3n^2+n)/n^3}{6}\right)\right)
\end{aligned}$$

Exercise 5.3.65 (continued 3)

Proof (continued).

$$\begin{aligned}\int_a^b x^2\,dx &= \lim_{n\to\infty}(b-a)\left(a^2+2a(b-a)\left(\frac{(n^2+n)/n^2}{2}\right)\right.\\ &\qquad\left.+(b-a)^2\left(\frac{(2n^3+3n^2+n)/n^3}{6}\right)\right)\\ &= \lim_{n\to\infty}(b-a)\left(a^2+2a(b-a)\left(\frac{1+1/n}{2}\right)\right.\\ &\qquad\left.+(b-a)^2\left(\frac{2+3/n+1/n^2}{6}\right)\right)\\ &= (b-a)\left(a^2+2a(b-a)\left(\frac{1+\lim_{n\to\infty}1/n}{2}\right)\right.\\ &\qquad\left.+(b-a)^2\left(\frac{2+3\lim_{n\to\infty}(1/n)+(\lim_{n\to\infty}1/n)^2}{6}\right)\right)\end{aligned}$$

Exercise 5.3.65 (continued 4)

Proof (continued).

$$\begin{aligned}\int_a^b x^2\,dx &= (b-a)\left(a^2 + 2a(b-a)\left(\frac{1+(0)}{2}\right)\right. \\ &\quad \left. +(b-a)^2\left(\frac{2+3(0)+(0)^2}{6}\right)\right) \\ &= (b-a)\left(a^2 + a(b-a) + (b-a)^2(1/3)\right) \\ &= (b-a)(a^2 + ab - a^2 + b^2/3 - 2ab/3 + a^2/3) \\ &= (b-a)(ab/3 + b^2/3 + a^2/3) \\ &= (ab^2 + b^3 + a^2b - a^2b - ab^2 - a^3)/3 \\ &= \frac{b^3}{3} - \frac{a^3}{3}. \qquad \square\end{aligned}$$

Exercise 5.3.36

Exercise 5.3.36. Use Equation (4) (see Exercise 5.3.65) to evaluate the integral $\int_0^{\pi/2} \theta^2 \, d\theta$.

Solution. The integrand is $f(\theta) = \theta^2$, the lower bound of the integral is $a = 0$, and the upper bound of the integral is $b = \pi/2$. So by Equation (4) (Exercise 5.3.65),

$$\int_0^{\pi/2} \theta^2 \, d\theta = \frac{b^3}{3} - \frac{a^3}{3} = \frac{(\pi/2)^3}{3} - \frac{0^3}{3} = \boxed{\frac{\pi^3}{24}}.$$

□

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$$\int_0^{\pi/2} \theta^2 \, d\theta = \frac{b^3}{3} - \frac{a^3}{3} = \frac{(\pi/2)^3}{3} - \frac{0^3}{3} = \boxed{\frac{\pi^3}{24}}.$$

□

Exercise 5.3.18

Exercise 5.3.18. Graph the integrand and use known area formulas to evaluate the integral: $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$.

Solution. Notice that with $y = \sqrt{16 - x^2}$, we have $y^2 = (\sqrt{16 - x^2})^2 = 16 - x^2$ and $y \geq 0$. So $x^2 + y^2 = 16$ and $y \geq 0$. So the graph of $y = \sqrt{16 - x^2}$ is the top half (since $y \geq 0$) of a circle of radius $r = 4$ and center $(0, 0)$:

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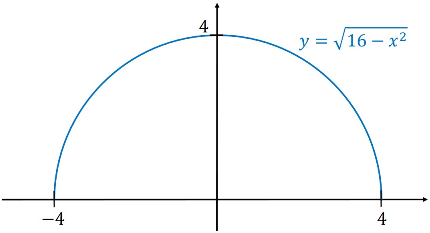

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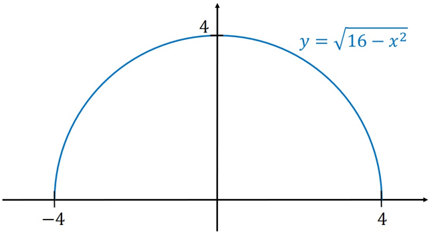
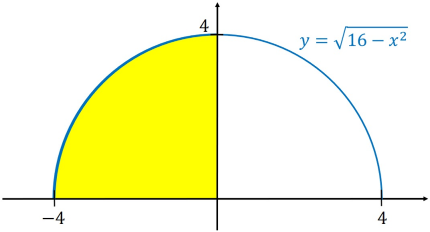


Exercise 5.3.18 (continued)

Solution.



Since $y = f(x) = \sqrt{16 - x^2}$ is non-negative, then (by definition) the definite integral $\int_{-4}^{0} \sqrt{16 - x^2}\, dx$ is the area under the curve $y = \sqrt{16 - x^2}$ (and above the x-axis) from $a = -4$ to $b = 0$. That is, the integral is $1/4$ of the area of a circle of radius $r = 4$. Therefore,

$$\int_{-4}^{0} \sqrt{16 - x^2}\, dx = \left. \frac{\pi(r)^2}{4} \right|_{r=4} = \frac{\pi(4)^2}{4} = \boxed{4\pi}. \quad \square$$

Exercise 5.3.62

Exercise 5.3.62. Graph the function $h(x) = |x|$ and find the average value over the intervals **(a)** $[-1, 0]$, **(b)** $[0, 1]$, and **(c)** $[-1, 1]$.

Solution. We consider the graph and relevant areas:

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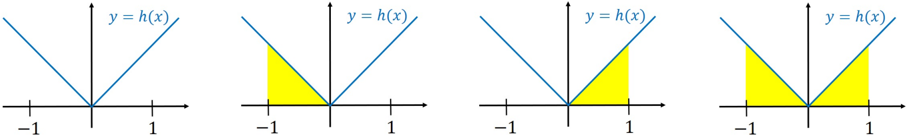

By definition $\text{av}(h) = \frac{1}{b-a}\int_a^b h(x)\,dx$, so calculating the integrals using areas we have

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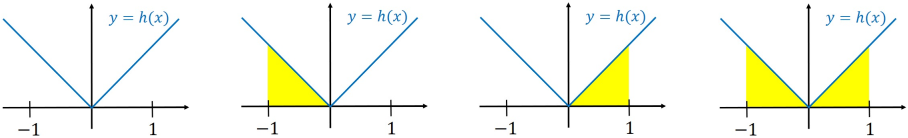

By definition $\text{av}(h) = \frac{1}{b-a} \int_a^b h(x)\, dx$, so calculating the integrals using areas we have

(a) The average of h over $[-1, 0]$ is $\frac{1}{(0)-(-1)} \int_{-1}^{0} h(x)\, dx = \boxed{1/2}$.

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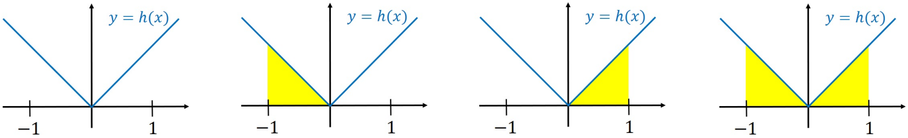

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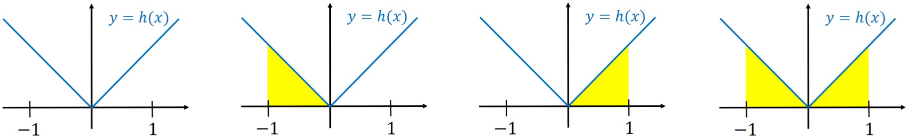

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(c) The average of h over $[-1, 1]$ is $\frac{1}{(1)-(-1)}\int_{-1}^{1} h(x)\,dx = 1/2(1) = \boxed{1/2}$. □

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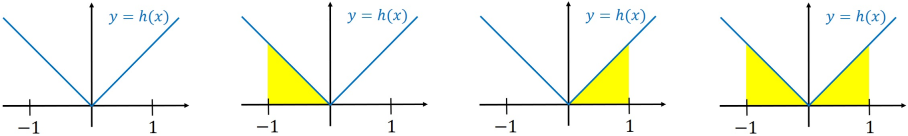

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$\frac{1}{(1)-(-1)}\int_{-1}^{1} h(x)\,dx = 1/2(1) = \boxed{1/2}$. □

Exercise 5.3.76

Exercise 5.3.76. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Solution. Let $f(x) = \sqrt{x+8} = (x+8)^{1/2}$. Then $f'(x) = \frac{1}{2}(x+8)^{-1/2} = \frac{1}{2\sqrt{x+8}}$ and so the only critical point of f is $x = -8$. So continuous function f has no critical points in $[0,1]$ and hence by the technique of Section 4.1, "Extreme Values of Functions on Closed Intervals," the extremes of f on $[0,1]$ occur at the endpoints.

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Exercise 5.3.76 (continued)

Exercise 5.3.76. Show that the value of $\int_0^1 \sqrt{x+8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

Solution (continued). By Theorem 5.2(6), the Max-Min Inequality, we have

$$\min f \cdot (b-a) = (2\sqrt{2})((1)-(0)) \leq \int_a^b f(x)\,dx$$

$$= \int_0^1 \sqrt{x+8}\,dx \leq \max f \cdot (b-a) = (3)((1)-(0)),$$

of $2\sqrt{2} \leq \int_0^1 \sqrt{x+8}\,dx \leq 3$, as claimed. $\square$

Exercise 5.3.88

Exercise 5.3.88. If you average 30 miles/hour on a 150 mile trip and then return over the same 150 miles at the rate of 50 miles/hour, what is your average speed for the trip? Give reasons for your answer.

Solution. We define function $f(t)$ as your speed as a function of time t, where t is measured in hours and f is measured in miles/hour. So we have f defined piecewise as $f(t) = 30$ miles/hour for t between 0 hours and 5 hours (since it takes 5 hours to travel 150 miles at 30 miles/hour) and $f(t) = 50$ miles/hour for t between 5 hours and 8 hours (since it takes 3 hours to travel 150 miles at 50 miles/hour): $f(t) = \begin{cases} 30, & 0 \leq t < 5 \\ 50, & 5 \leq t \leq 8 \end{cases}$

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Solution. We define function $f(t)$ as your speed as a function of time t, where t is measured in hours and f is measured in miles/hour. So we have f defined piecewise as $f(t) = 30$ miles/hour for t between 0 hours and 5 hours (since it takes 5 hours to travel 150 miles at 30 miles/hour) and $f(t) = 50$ miles/hour for t between 5 hours and 8 hours (since it takes 3 hours to travel 150 miles at 50 miles/hour): $f(t) = \begin{cases} 30, & 0 \leq t < 5 \\ 50, & 5 \leq t \leq 8 \end{cases}$

Exercise 5.3.88 (continued)

Solution (continued). $\ldots f(t) = \begin{cases} 30, & 0 \le t < 5 \\ 50, & 5 \le t \le 8 \end{cases}$ So, by definition, the average speed (i.e., the average of f over $[0,8]$) is

$$\begin{aligned} \text{av}(f) &= \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{(8)-(0)}\int_0^8 f(t)\,dt \\ &= \frac{1}{8}\left(\int_0^5 f(t)\,dt + \int_5^8 f(t)\,dt\right) \text{ by Theorem 5.2(5), Additivity} \\ &= \frac{1}{8}\left(\int_0^5 30\,dt + \int_5^8 50\,dt\right) \\ &= \frac{1}{8}((30)((5)-(0)) + (50)((8)-(5))) \text{ by Exercise 5.3.63} \\ &= \frac{1}{8}(150+150) = \frac{300}{8} = \boxed{\frac{75}{2}\text{ miles/hour.}} \end{aligned}$$

□

Exercise 5.3.88 (continued)

Solution (continued). $\ldots f(t) = \begin{cases} 30, & 0 \le t < 5 \\ 50, & 5 \le t \le 8 \end{cases}$ So, by definition, the average speed (i.e., the average of f over $[0, 8]$) is

$$\begin{aligned} \text{av}(f) &= \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{(8)-(0)}\int_0^8 f(t)\,dt \\ &= \frac{1}{8}\left(\int_0^5 f(t)\,dt + \int_5^8 f(t)\,dt\right) \text{ by Theorem 5.2(5), Additivity} \\ &= \frac{1}{8}\left(\int_0^5 30\,dt + \int_5^8 50\,dt\right) \\ &= \frac{1}{8}((30)((5)-(0)) + (50)((8)-(5))) \text{ by Exercise 5.3.63} \\ &= \frac{1}{8}(150+150) = \frac{300}{8} = \boxed{\frac{75}{2} \text{ miles/hour.}} \end{aligned}$$

□