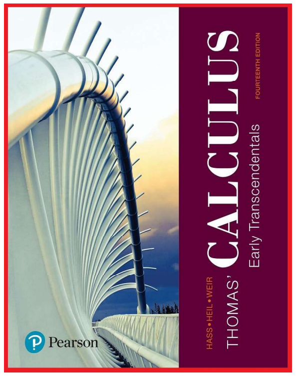
Calculus 1

Chapter 5. Integrals

5.4. The Fundamental Theorem of Calculus—Examples and Proofs



Theorem 5.3

Theorem 5.3. The Mean Value Theorem for Definite Integrals.
If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. By the Max-Min Inequality (Theorem 5.2(6)), we have

$$\min f \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \max f.$$

Since f is continuous, f must assume any value between $\min f$ and $\max f$, including $\frac{1}{b-a}\int_a^b f(x)\,dx$ by the Intermediate Value Theorem (Theorem 2.11). □

Example 5.4.1

Example 5.4.1. Prove that if f is continuous on $[a, b]$, $a \ne b$, and if $\int_a^b f(x)\,dx = 0$, then $f(x) = 0$ at least once in $[a, b]$.

Proof. Since f is continuous on $[a, b]$, then by The Mean Value Theorem for Definite Integrals (Theorem 5.3) we have $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$ for some $c \in [a, b]$. Since we are given that $\int_a^b f(x)\,dx = 0$, then for this value c we have

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}(0) = 0,$$

so that $f(c) = 0$, as claimed. □

Theorem 5.4(a)

Theorem 5.4(a). The Fundamental Theorem of Calculus, Part 1.
If f is continuous on $[a, b]$ then the function

$$F(x) = \int_a^x f(t)\,dt$$

has a derivative at every point x in $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Proof. Notice that by Additivity, Theorem 5.2(5),

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

So

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}[F(x+h) - F(x)] = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Theorem 5.4(a) (continued)

Proof (continued). Since f is continuous, The Mean Value Theorem for Definite Integrals (Theorem 5.3) implies that for some $c \in [x, x+h]$ we have

$$f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since $c \in [x, x+h]$, then $\lim_{h\to 0} f(c) = f(x)$ (since f is continuous at x). Therefore

$$\begin{aligned}\frac{dF}{dx} &= \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt\\ &= \lim_{h\to 0} f(c) = f(x)\end{aligned}$$

□

Exercise 5.4.46

Exercise 5.4.46 Find dy/dx when $y = \int_1^x \frac{1}{t}\,dt$ where $x > 0$.

Solution. Since $f(t) = 1/t$ is continuous on interval $[x, 1]$ when $0 < x < 1$ and is continuous on interval $[1, x]$ when $1 < x$, then by The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)), we have

$$\frac{d}{dx}[y] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x},$$

or $\boxed{\frac{dy}{dx} = \frac{1}{x}}$. □

Exercise 5.4.48

Exercise 5.4.48 Find dy/dx when $y = x\int_2^{x^2} \sin(t^3)\,dt$.

Solution. First, we let $u = u(x) = x^2$ so that y is in the form $y = x\int_2^u \sin(t^3)\,dt$. Then by the Derivative Product Rule (Theorem 3.3.G), The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)), and the Chain Rule (Theorem 3.2)

$$\begin{aligned}\frac{d}{dx}[y] &= \frac{d}{dx}\left[x\int_2^u \sin(t^3)\,dt\right]\\ &= [1]\left(\int_2^u \sin(t^3)\,dt\right) + (x)\frac{d}{dx}\left[\int_2^u \sin(t^3)\,dt\right]\\ &= [1]\left(\int_2^u \sin(t^3)\,dt\right) + (x)\frac{d}{du}\left[\int_2^u \sin(t^3)\,dt\right]^{\frown}\left[\frac{du}{dx}\right]\end{aligned}$$

Exercise 5.4.48 (continued)

Exercise 5.4.48 Find dy/dx when $y = x\int_2^{x^2} \sin(t^3)\,dt$.

Solution (continued). ...

$$\begin{aligned}\frac{d}{dx}[y] &= [1]\left(\int_2^u \sin(t^3)\,dt\right) + (x)\frac{d}{du}\left[\int_2^u \sin(t^3)\,dt\right]^{\frown}\left[\frac{du}{dx}\right]\\ &= \left(\int_2^u \sin(t^3)\,dt\right) + (x)\left[\sin((u)^3)^{\frown}\left[\frac{du}{dx}\right]\right]\\ &= \left(\int_2^u \sin(t^3)\,dt\right) + (x)[\sin((x^2)^3)^{\frown}[2x]]\\ &= \boxed{\int_2^{x^2} \sin(t^3)\,dt + 2x^2\sin(x^6)}.\end{aligned}$$

□

Exercise 5.4.54

Exercise 5.4.54 Find dy/dx when $y = \int_{2^x}^{1} \sqrt[3]{t}\, dt$.

Solution. Then by the Derivative Product Rule (Theorem 3.3.G) and The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)),

$$\begin{aligned} \frac{d}{dx}[y] &= \frac{d}{dx}\left[\int_{2^x}^{1} \sqrt[3]{t}\, dt\right] = \frac{d}{dx}\left[-\int_{1}^{2^x} \sqrt[3]{t}\, dt\right] \text{ by Theorem 5.2(1)} \\ &= -\frac{d}{dx}\left[\int_{1}^{u} \sqrt[3]{t}\, dt\right] \text{ where } u = 2^x \\ &= -\frac{d}{du}\left[\int_{1}^{u} \sqrt[3]{t}\, dt\right]\left[\frac{du}{dx}\right] \text{ by the Chain Rule, Theorem 3.2} \\ &= -\sqrt[3]{u}\left[\frac{du}{dx}\right] = -\sqrt[3]{2^x}[(\ln 2)2^x] = -\ln 2\sqrt[3]{2^x}2^x = \boxed{-(\ln 2)2^{4x/3}}. \end{aligned}$$

□

Theorem 5.4(b)

Theorem 5.4(b). The Fundamental Theorem of Calculus, Part 2. If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof. We know from the first part of the Fundamental Theorem (Theorem 5.4(a)) that

$$G(x) = \int_a^x f(t)\, dt$$

defines *an* antiderivative of f. Therefore if F is *any* antiderivative of f, then $F(x) = G(x) + k$ for some constant k by Corollary 4.2 ("Functions with the Same Derivative Differ by a Constant").

Theorem 5.4(b) (continued)

Theorem 5.4(b). The Fundamental Theorem of Calculus, Part 2. If f is continuous at every point of $[a, b]$ and if F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof (continued). Therefore

$$\begin{aligned} F(b) - F(a) &= [G(b) + k] - [G(a) + k] = G(b) - G(a) \\ &= \int_a^b f(t)\, dt - \int_a^a f(t)\, dt = \int_a^b f(t)\, dt - 0 \\ &= \int_a^b f(t)\, dt, \end{aligned}$$

as claimed. □

Exercise 5.4.6

Exercise 5.4.6. Evaluate the integral $\int_{-2}^{2} (x^3 - 2x + 3)\, dx$.

Solution. By The Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)), we just need an antiderivative F of the integrand $f(x) = x^3 - 2x + 3$. We can take $F(x) = x^4/4 - x^2 + 3x$. Then we have

$$\int_{-2}^{2} (x^3 - 2x + 3)\, dx = \left(\frac{x^4}{4} - x^2 + 3x\right)\Big|_{-2}^{2}$$

$$= \left(\frac{(2)^4}{4} - (2)^2 + 3(2)\right) - \left(\frac{(-2)^4}{4} - (-2)^2 + 3(-2)\right)$$

$$= 4 - 4 + 6 - 4 + 4 + 6 = \boxed{12}.$$ □

Exercise 5.4.14

Exercise 5.4.14. Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sin^2 t\, dt$. HINT: By a half-angle formula, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

Solution. By The Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)), we just need an antiderivative F of the integrand $f(t) = \sin^2 t$. Since $\sin^2 t = \dfrac{1-\cos 2t}{2} = \dfrac{1}{2}(1-\cos 2t)$, we can take $F(t) = \dfrac{1}{2}\left(t - \dfrac{\sin 2t}{2}\right)$ (see Table 4.2 entry 3 in Section 4.8). Then we have

$$\int_{-\pi/3}^{\pi/3} \sin^2 t\, dt = \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Bigg|_{-\pi/3}^{\pi/3}$$

Exercise 5.4.14 (continued)

Exercise 5.4.14. Evaluate the integral $\int_{-\pi/3}^{\pi/3} \sin^2 t\, dt$. HINT: By a half-angle formula, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$.

Solution (continued). ...

$$\int_{-\pi/3}^{\pi/3} \sin^2 t\, dt = \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)\Bigg|_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3}\right) - \frac{\sin 2(\pi/3)}{2}\right) - \frac{1}{2}\left(\left(\frac{-\pi}{3}\right) - \frac{\sin 2(-\pi/3)}{2}\right)$$

$$= \frac{\pi}{6} - \frac{(\sqrt{3}/2)}{4} + \frac{\pi}{6} - \frac{(\sqrt{3}/2)}{4} = \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}. \quad \square$$

Exercise 5.4.22

Exercise 5.4.22. Evaluate the integral $\int_{-3}^{-1} \dfrac{y^5 - 2y}{y^3}\, dy$.

Solution. We apply The Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)). We modify the integrand first so that find an antiderivative. We have

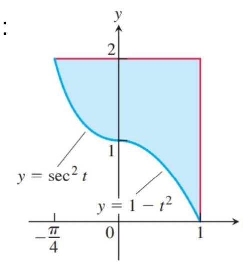
$$\int_{-3}^{-1} \frac{y^5-2y}{y^3}\, dy = \int_{-3}^{-1} y^2 - 2y^{-2}\, dy = \left(\frac{y^3}{3} - 2(-y^{-1})\right)\Bigg|_{-3}^{-1}$$

$$= \left(\frac{y^3}{3} + \frac{2}{y}\right)\Bigg|_{-3}^{-1} = \left(\frac{(-1)^3}{3} + \frac{2}{(-1)}\right) - \left(\frac{(-3)^3}{3} + \frac{2}{(-3)}\right)$$

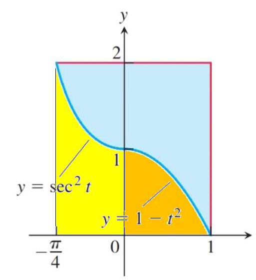
$$= \left(-\frac{1}{3} - 2\right) - \left(-9 - \frac{2}{3}\right) = 7 + \frac{1}{3} = \boxed{\frac{22}{3}}. \quad \square$$

Exercise 5.4.64

Exercise 5.4.64. Find the area of the shaded region:



Solution. We know that a definite integral over $[a, b]$ of a nonnegative function f is (by definition) the area under $y = f(x)$ from a to b. Notice that the desired area (in blue) is the area in a rectangle of width $1 + \pi/4$ and height 2 minus the area under $y = \sec^2 t$ from $-\pi/4$ to 0 (in yellow) and minus the area under $y = 1 - t^2$ from 0 to 1 (in orange):



That is, the desired area is

$$(1+\pi/4)(2) - \int_{-\pi/4}^{0} \sec^2 t\, dt - \int_{0}^{1} 1 - t^2\, dt.$$

Exercise 5.4.64 (continued)

Solution (continued). . . . the desired area is

$$(1+\pi/4)(2) - \int_{-\pi/4}^{0} \sec^2 t\, dt - \int_0^1 1 - t^2\, dt$$

$$= 2 + \pi/2 - \tan t|_{-\pi/4}^{0} - (t - t^3/3)|_0^1$$

$$= 2 + \pi/2 - (\tan(0) - \tan(-\pi/4)) - (((1) - (1)^3/3) - ((0) - (0)^3/3))$$

$$= 2 + \pi/2 - (1) - (2/3) = \boxed{1/3 + \pi/2}. \qquad \square$$

Exercise 5.4.82

Exercise 5.4.82. Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)\, dt$ at $x = -1$.

Solution. Recall that the linearization of g at $x = a$ is $L(x) = g(a) + g'(a)(x-a)$. We have

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left[3 + \int_1^{x^2} \sec(t-1)\, dt\right] \\ &= \frac{d}{du}\left[3 + \int_1^{u} \sec(t-1)\, dt\right]\frac{du}{dx} \text{ by the Chain Rule, where } u = x^2 \\ &= 0 + \sec(u-1)\frac{du}{dx} \text{ by The Fundamental Theorem of Calculus,} \\ &\quad \text{Part 1 (Theorem 5.4(a))} \\ &= \sec(x^2-1)[2x] = 2x\sec(x^2-1). \end{aligned}$$

Exercise 5.4.82 (continued)

Exercise 5.4.82. Find the linearization of $g(x) = 3 + \int_1^{x^2} \sec(t-1)\, dt$ at $x = -1$.

Solution (continued). With $g(x) = 3 + \int_1^{x^2} \sec(t-1)\, dt$ and $g'(x) = 2x\sec^2(x^2-1)$, we have $g(a) = g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\, dt = 3 + 0 = 3$ and $g'(a) = g'(-1) = 2(-1)\sec((-1)^2-1) = -2\sec(0) = -2(1) = -2$. So the linearization of g at $x = a = -1$ is $L(x) = g(-1) + g'(-1)(x-(-1))$ is

$$L(x) = (3) + (-2)(x - (-1)) = 3 - 2x - 2 = \boxed{-2x+1}.$$

$\square$

Exercise 5.4.72

Exercise 5.4.72. Find a function f satisfying the equation $f(x) = e^2 + \int_1^x f(t)\, dt$.

Solution. First, we differentiation with respect to x to get

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[e^2 + \int_1^x f(t)\, dt\right] = f(x)$$

by The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)). So $f'(x) = f(x)$. Some functions satisfying this condition are functions of the form ke^x where k is some constant. Notice also that $f(1) = e^2 + \int_1^{(1)} f(t)\, dt = e^2 + 0 = e^2$. Now $(ke^x)|_{x=1} = ke^{(1)} = ke$, so with $k = e$ we have $f(x) = ee^x = e^{x+1}$.

Exercise 5.4.72 (continued)

Exercise 5.4.72. Find a function f satisfying the equation $f(x) = e^2 + \int_1^x f(t)\,dt$.

Solution (continued). With $f(x) = e^{x+1}$, we have that both $f(1) = e^{(1)+1} = e^2$ and (by the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)):

$$e^2 + \int_1^x f(t)\,dt = e^2 + \int_1^x e^{t+1}\,dt = e^2 + e^{t+1}\Big|_{t=1}^{t=x}$$

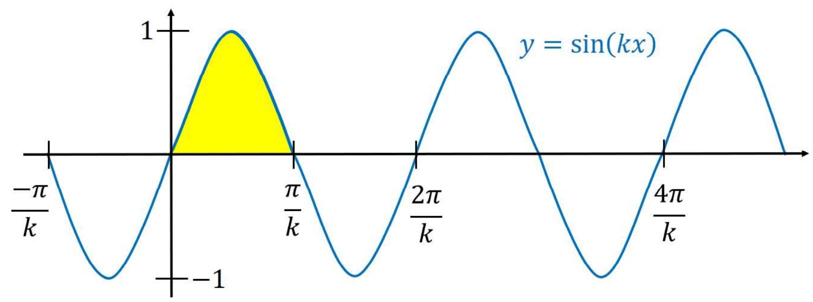
$$= e^2 + (e^{(x)+1} - e^{(1)+1}) = e^2 + e^{x+1} - e^2 = e^{x+1} = f(x),$$

as desired. So one such function is $\boxed{f(x) = e^{x+1}}$. □

Exercise 5.4.74

Exercise 5.4.74. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

Solution. The graph of $y = \sin kx$, along with the area under one arch, is:



So the area is $A = \int_0^{\pi/k} \sin kx\,dx$ (since $\sin kx \geq 0$ for $x \in [0, \pi/k]$).

Exercise 5.4.74 (continued)

Exercise 5.4.74. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

Solution (continued). ...So the area is $A = \int_0^{\pi/k} \sin kx\,dx$ (since $\sin kx \geq 0$ for $x \in [0, \pi/k]$). Evaluating the integral using the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)) we have

$$A = \int_0^{\pi/k} \sin kx\,dx = \frac{-\cos kx}{k}\Big|_0^{\pi/k} = \frac{-\cos k(\pi/k)}{k} - \frac{-\cos k(0)}{k}$$

$$= \frac{-\cos \pi}{k} + \frac{\cos 0}{k} = \frac{-(-1)}{k} + \frac{1}{k} = \boxed{\frac{2}{k}},$$

as claimed (where the antiderivative of $\sin kx$ is given by Table 4.2(2) in Section 4.8). □

Example 5.4.8

Example 5.4.8. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.

Solution. We need the sign of $f(x) = x^3 - x^2 - 2x$ so that we can separate the region bounded by the x-axis and the graph of $y = f(x)$ into a part where the function is positive and a part where the function is negative. Notice that

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2)$$

so that $f(x) = 0$ for $x = -1$, $x = 0$, and $x = 2$. Since f is continuous (it is a polynomial function), then we perform a sign test of f as we did when applying the First and Second Derivative Tests in Chapter 4.

Example 5.4.8 (continued 1)

Example 5.4.8. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.
Solution (continued). Consider:

interval	$(-\infty, -1)$	$(-1, 0)$
test value k	-2	$-1/2$
$f(k)$	$(-2)^3 - (-2)^2 - 2(-2) = -8$	$(-1/2)^3 - (-1/2)^2 - 2(-1/2) = 5/8$
$f(x)$	$-$	$+$

interval	$(0, 2)$	$(2, \infty)$
test value k	1	3
$f(k)$	$(1)^3 - (1)^2 - 2(1) = -2$	$(3)^3 - (3)^2 - 2(3) = 12$
$f(x)$	$-$	$+$

So $f(x) \ge 0$ for $x \in [-1, 0] \cup [2, \infty)$, and $f(x) \le 0$ for $x \in (-\infty, -1] \cup [0, 2]$. In particular, on $[-1, 0]$ we have $f(x) \ge 0$ (and the area between f and the x-axis is given by the integral of f over $[-1, 0]$), and on $[0, 2]$ we have $f(x) \le 0$ (and the *negative* of the area between f and the x-axis is given by the integral of f over $[0, 2]$).

Example 5.4.8 (continued 2)

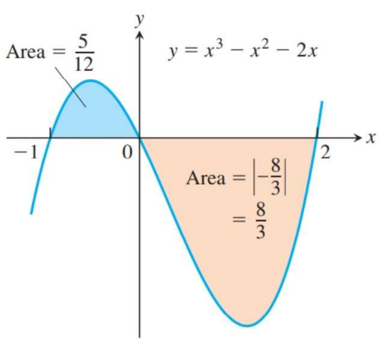
Solution (continued). So the desired area is

$$\begin{aligned}
A &= \int_{-1}^{0} f(x)\,dx + \left(-\int_{0}^{2} f(x)\,dx\right) \\
&= \int_{-1}^{0} x^3 - x^2 - 2x\,dx - \int_{0}^{2} x^3 - x^2 - 2x\,dx \\
&= \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{-1}^{0} - \left(\frac{x^4}{4} - \frac{x^3}{3} - x^2\right)\Big|_{0}^{2} \\
&= \left(\frac{(0)^4}{4} - \frac{(0)^3}{3} - (0)^2\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right) \\
&\quad - \left(\left(\frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2\right) - \left(\frac{(0)^4}{4} - \frac{(0)^3}{3} - (0)^2\right)\right) \\
&= ((0) - (1/4 + 1/3 - 1)) - ((4 - 8/3 - 4) - (0)) \\
&= 5/12 - (-8/3) = 5/12 + 8/3 = \boxed{37/12}.
\end{aligned}$$

Example 5.4.8 (continued 3)

Example 5.4.8. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution (continued). . . . So the desired area is $A = 5/12 - (-8/3) = 5/12 + 8/3 = 37/12$. The text book gives the following graph to illustrate how the area is calculated:



□