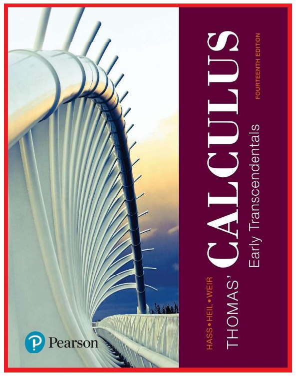
Calculus 1

Chapter 5. Integrals

5.5. Indefinite Integrals and the Substitution Method—Examples and Proofs



Exercise 5.5.20

Exercise 5.5.20. Evaluate: $\int 3y\sqrt{7-3y^2}\,dy$.

Solution. We let $f(y) = 7-3y^2$ so that $f'(y) = -6y$. We then have

$$\int 3y\sqrt{7-3y^2}\,dy = \frac{1}{-2}\int -6y(7-3y^2)^{1/2}\,dy = \frac{-1}{2}\int (f(y))^{1/2}f'(y)\,dy$$

$$= \frac{-1}{2}\left(\frac{1}{3/2}(f(y))^{3/2}\right) + C = \boxed{\frac{-1}{3}(7-3y^2)^{3/2} + C}.$$

Alternatively, let $u = 7-3y^2$ so that $du = -6y\,dy$ or $\frac{-1}{2}\,du = 3y\,dy$. We then have

$$\int 3y\sqrt{7-3y^2}\,dy = \int \sqrt{u}\frac{-1}{2}\,du = \frac{-1}{2}\int u^{1/2}\,du = \frac{-1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{-1}{3}(7-3y^2)^{3/2} + C. \qquad \square$$

Theorem 5.6

Theorem 5.6. The Substitution Rule. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. By the Chain Rule (Theorem 3.2), $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$ for F an antiderivative of f because

$$\frac{d}{dx}[F(g(x))] = F'(g(x))[g'(x)] = f(g(x))g'(x).$$

With $u = u(x) = g(x)$, then

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}[F(g(x))]\,dx = F(g(x)) + C \text{ by Theorem 4.8}$$

$$= F(u) + C = \int F'(u)\,du \text{ by Theorem 4.8}$$

$$= \int f(u)\,du, \text{ as claimed.} \qquad \square$$

Exercise 5.5.6

Exercise 5.5.6. Evaluate: $\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx$.

Solution. We let u be some function of x where we see a multiple of u' as part of the integrand. We choose $u = 1+\sqrt{x} = 1+x^{1/2}$ so that $du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$ or $2\,du = \frac{1}{\sqrt{x}}\,dx$. Then

$$\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx = \int (1+\sqrt{x})^{1/3}\frac{1}{\sqrt{x}}\,dx = \int u^{1/3}2\,du$$

$$= 2\int u^{1/3}\,du = 2\left(\frac{3}{4}u^{4/3}\right) + C = \boxed{\frac{3}{2}(1+\sqrt{x})^{4/3} + C}. \qquad \square$$

Exercise 5.5.32

Exercise 5.5.32. Evaluate: $\displaystyle\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz$.

Solution. We let u be some function of z where we see a multiple of u' as part of the integrand. We choose $u = \sec z$ so that $du = \sec z \tan z\,dz$. Then

$$\int \frac{\sec z \tan z}{\sqrt{\sec z}}\,dz = \int \frac{1}{(\sec z)^{1/2}} \sec z \tan z\,dz = \int (\sec z)^{-1/2} \sec z \tan z\,dz$$

$$= \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{u} + C = \boxed{2\sqrt{\sec z} + C}. \qquad \square$$

Exercise 5.5.56

Exercise 5.5.56. Evaluate: $\displaystyle\int \frac{\ln\sqrt{t}}{t}\,dt$.

Solution. First we rewrite the integral as $\displaystyle\int \frac{\ln\sqrt{t}}{t}\,dt = \int \frac{\ln t^{1/2}}{t}\,dt = \frac{1}{2}\int \frac{\ln t}{t}\,dt$. We now let u be some function of t where we see a multiple of u' as part of the integrand. We choose $u = \ln t$ so that $du = \dfrac{1}{t}\,dt$. Then

$$\int \frac{\ln\sqrt{t}}{t}\,dt = \frac{1}{2}\int \frac{\ln t}{t}\,dt = \frac{1}{2}\int \ln t \frac{1}{t}\,dt$$

$$= \frac{1}{2}\int u\,du = \frac{1}{2}\left(\frac{1}{2}u^2\right) + C = \boxed{\frac{1}{4}(\ln t)^2 + C}. \qquad \square$$

Exercise 5.5.60

Exercise 5.5.60. Evaluate: $\displaystyle\int \frac{1}{\sqrt{e^{2\theta} - 1}}\,d\theta$.

Solution. We want to let u be some function of θ where we see a multiple of u' as part of the integrand. There appears to be no obvious such choice for u. Notice from Table 4.2 (or Table 4.2.A) from Section 4.8, the integral most closely resembles entry #12 (from Table 4.2.A): $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + C$. So we try $u = e^{\theta}$ and $du = e^{\theta}\,d\theta$. We then have $\dfrac{du}{e^{\theta}} = d\theta$ or $\dfrac{du}{u} = d\theta$. Then

$$\int \frac{1}{\sqrt{e^{2\theta} - 1}}\,d\theta = \int \frac{1}{\sqrt{(e^{\theta})^2 - 1}}\,d\theta = \int \frac{1}{\sqrt{u^2-1}}\,\frac{du}{u}$$

$$= \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}(u) + C = \boxed{\sec^{-1}(e^{\theta}) + C}. \qquad \square$$

Example 5.5.7(c)

Example 5.5.7(c). Evaluate: $\displaystyle\int \tan x\,dx$.

Solution. First we rewrite the integral as $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$. We now let u be some function of x where we see a multiple of u' as part of the integrand. We choose $u = \cos x$ so that $du = -\sin x\,dx$ or $-du = \sin x\,dx$. Then

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{\cos x}\sin x\,dx = \int \frac{1}{u}(-du)$$

$$= -\ln|u| + C = -\ln|\cos x| + C = \ln|(\cos x)^{-1}| + C = \boxed{\ln|\sec x| + C}. \qquad \square$$

Example 5.5.8(b)

Example 5.5.8(b). Evaluate: $\displaystyle\int \sec x\,dx$.

Solution. This one requires a trick. We rewrite the integral as

$$\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx.$$

Now we let $u = \sec x + \tan x$ so that $du = (\sec x \tan x + \sec^2 x)\,dx$. Then

$$\int \sec x\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)\,dx$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \boxed{\ln|\sec x + \tan x| + C}. \qquad \square$$

Exercise 5.5.68

Exercise 5.5.68. Evaluate: $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$: **(a)** by first letting $u = x-1$, followed by $v = \sin u$, and then $w = 1+v^2$, **(b)** by first letting $u = \sin(x-1)$ followed by $v = 1+u^2$, and **(c)** by letting $u = 1+\sin^2(x-1)$.

Solution. **(a)** Following the instructions, we let $u = x-1$ so that $du = dx$. Then $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx =$ $\int \sqrt{1+\sin^2(u)}\,\sin(u)\cos(u)\,du$. Next, we let $v = \sin u$ so that $dv = \cos u\,du$. Then $\int \sqrt{1+\sin^2(u)}\,\sin(u)\cos(u)\,du = \int \sqrt{1+v^2}\,v\,dv$. Finally, we let $w = 1+v^2$ so that $dw = 2v\,dv$ or $dw/2 = v\,dv$. Then

$$\int \sqrt{1+v^2}\,v\,dv = \int \sqrt{w}\,dw/2 = \frac{1}{2}\int w^{1/2}\,dw = \frac{1}{2}\left(\frac{2}{3}w^{3/2}\right) + C$$

$$= \frac{1}{3}(1+v^2)^{3/2} + C = \frac{1}{3}(1+(\sin u)^2)^{3/2} + C = \boxed{\frac{1}{3}(1+\sin^2(x-1))^{3/2} + C}. \square$$

Exercise 5.5.68 (continued 1)

Exercise 5.5.68. Evaluate: $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$: **(b)** by first letting $u = \sin(x-1)$ followed by $v = 1+u^2$, and **(c)** by letting $u = 1+\sin^2(x-1)$.

Solution. **(b)** Following the instructions, we let $u = \sin(x-1)$ so that $du = \cos(x-1)\,dx$. Then $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \sqrt{1+u^2}\,u\,du$. Next, we let $v = 1+u^2$ so that $dv = 2u\,du$ or $dv/2 = u\,du$. Then

$$\int \sqrt{1+u^2}\,u\,du = \int \sqrt{v}\,dv/2 = \frac{1}{2}\int v^{1/2}\,dv = \frac{1}{2}\left(\frac{2}{3}v^{3/2}\right) + C$$

$$= \frac{1}{3}(1+u^2)^{3/2} + C = \boxed{\frac{1}{3}(1+\sin^2(x-1))^{3/2} + C.} \qquad \square$$

Exercise 5.5.68 (continued 2)

Exercise 5.5.68. Evaluate: $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$: **(c)** by letting $u = 1+\sin^2(x-1)$.

Solution. **(c)** Following the instructions, we let $u = 1+\sin^2(x-1)$ so that $du = 2\sin(x-1)[\cos(x-1)[1]]\,dx$ or $du/2 = \sin(x-1)\cos(x-1)\,dx$. Then

$$\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx = \int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C = \boxed{\frac{1}{3}(1+\sin^2(x-1))^{3/2} + C}. \qquad \square$$

Exercise 5.5.70

Exercise 5.5.70. Solve the initial value problem: $\dfrac{d^2y}{dx^2} = 4\sec^2 2x \tan 2x$, $y'(0) = 4$, $y(0) = -1$.

Solution. First, $\dfrac{dy}{dx} \in \displaystyle\int \frac{d^2y}{dx^2}\,dx = \int 4\sec^2 2x \tan 2x\,dx$. With $u = \sec 2x$ we have $du = \sec 2x \tan 2x[2]\,dx$ or $du/2 = \sec 2x \tan 2x\,dx$. So

$$\int 4\sec^2 2x \tan 2x\,dx = 4\int \sec 2x \sec 2x \tan 2x\,dx$$

$$= 4\int u\frac{du}{2} = 2\int u\,du = 2\left(\frac{1}{2}u^2\right) + C = u^2 + C = \sec^2(2x) + C.$$

So $\dfrac{dy}{dx} = y' = \sec^2(2x) + k_1$ for some constant k_1. Since $y'(0) = 4$ then $y'(0) = \sec^2(2(0)) + k_1 = \sec^2(0) + k_1 = 1 + k_1 = 4$, or $k_1 = 3$. Hence $dy/dx = \sec^2(2x) + 3$.

Exercise 5.5.70 (continued)

Solution (continued). Next, $y \in \displaystyle\int \frac{dy}{dx}\,dx = \int \sec^2(2x) + 3\,dx$. With $u = 2x$ we have $du = 2\,dx$ or $du/2 = dx$. Then

$$\int \sec^2(2x) + 3\,dx = \int (\sec^2(u) + 3)\frac{du}{2} = \frac{1}{2}\int (\sec^2(u) + 3)\,du$$

$$= \frac{1}{2}(\tan u + 3u) + C = \frac{1}{2}\tan u + \frac{3}{2}u + C$$

$$= \frac{1}{2}\tan(2x) + \frac{3}{2}(2x) + C = \frac{1}{2}\tan(2x) + 3x + C.$$

So $y = \frac{1}{2}\tan(2x) + 3x + k_2$ for some constant k_2. Since $y(0) = -1$ then $y(0) = \frac{1}{2}\tan(2(0)) + 3(0) + k_2 = -1$, or $k_2 = -1$. Hence $\boxed{y = \dfrac{1}{2}\tan(2x) + 3x - 1}$. □