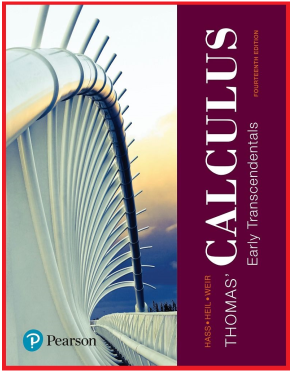
Calculus 1

Chapter 5. Integrals

5.6. Substitution and Area Between Curves—Examples and Proofs



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If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Proof. Let F be an antiderivative of f. Then $\frac{d}{dx}[F(g(x))] = F'(g(x))\overset{\curvearrowright}{[}g'(x)] = f(g(x))g'(x)$, so that $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. So

$$\begin{aligned}\int_a^b f(g(x))g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \text{ by the Fundamental Theorem} \\ &\quad \text{of Calculus, Part 2 (Theorem 5.4(b)} \\ &= F(g(b)) - F(g(a)) = F(u)\Big|_{u=g(a)}^{u=g(b)} \text{ with } u = g(x)\end{aligned}$$

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Proof (continued). ...

$$\begin{aligned}\int_a^b f(g(x))g(x)\,dx &= F(u)\Big|_{u=g(a)}^{u=g(b)} \text{ with } u = g(x)\\ &= \int_{g(a)}^{g(b)} f(u)\,du \text{ with } u = g(x) \text{ and by the}\\ &\quad\text{Fundamental Theorem of Calculus,}\\ &\quad\text{Part 2 (Theorem 5.4(b).}\end{aligned}$$

□

Exercise 5.6.22

Exercise 5.6.22. Evaluate $\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2}(y^2 + 4y - 4)\, dy$.

Solution. We apply Theorem 5.7 (Substitution in Definite Integrals) and let $u = g(y) = y^3 + 6y^2 - 12y + 9$. Then $g'(y) = 3y^2 + 12y - 12 = 3(y^2 + 4y - 4)$. Notice that f and g' are continuous everywhere, so the hypotheses of Theorem 5.7 are satisfied.

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$$\int_a^b f(g(y))g'(y)\,dy = \int_{g(a)}^{g(b)} f(u)\,du \text{ or}\ldots$$

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Exercise 5.6.22 (continued)

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Solution (continued). ...

$$\int_a^b f(g(y))g'(y)\,dy = \int_{g(a)}^{g(b)} f(u)\,du \text{ or}$$

$$\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\,dy$$

$$= \frac{1}{3}\int_0^1 (y^3+6y^2-12y+9)^{-1/2}3(y^2+4y-4)\,dy$$

$$= \frac{1}{3}\int_9^4 u^{-1/2}\,du = \frac{1}{3}2u^{1/2}\Big|_9^4 = \frac{1}{3}2\sqrt{(4)} - \frac{1}{3}2\sqrt{(9)} = \frac{4}{3} - 2 = \boxed{-\frac{2}{3}}. \quad \square$$

Exercise 5.6.18

Exercise 5.6.18. Evaluate $\displaystyle\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)\,d\theta$.

Solution. We have seen $\sec x$ and $\tan x$ "travel together" through this world of differentiation and antiderivatives (as have $\csc x$ and $\cot x$). Since $\cot x = 1/\tan x$, we start by modifying the integrand as

$$\cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) = \frac{\sec^2(\theta/6)}{\tan^5(\theta/6)}.$$

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We apply Theorem 5.7 (Substitution in Definite Integrals), let $f(u) = 1/u^5$, and let $u = g(\theta) = \tan(\theta/6)$. Then $g'(\theta) = \sec^2(\theta/\overset{\curvearrowright}{6})[1/6]$. Notice that f and g' are continuous everywhere, so the hypotheses of Theorem 5.7 are satisfied.

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Exercise 5.6.18. Evaluate $\displaystyle\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)\,d\theta$.

Solution (continued). ...

$$\int_a^b f(g(\theta))g'(\theta)\,d\theta = \int_{g(a)}^{g(b)} f(u)\,du \text{ or}$$

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)\,d\theta = 6\int_{\pi}^{3\pi/2} \frac{\sec^2(\theta/6)/6}{\tan^5(\theta/6)}\,d\theta = 6\int_{1/\sqrt{3}}^{1} \frac{1}{u^5}\,du$$

$$= 6\int_{1/\sqrt{3}}^{1} u^{-5}\,du = 6\frac{u^{-4}}{-4}\bigg|_{1/\sqrt{3}}^{1} = 6\frac{(1)^{-4}}{-4} - 6\frac{(1/\sqrt{3})^{-4}}{-4}$$

$$= 6\frac{-1}{4} + 6\frac{(\sqrt{3})^4}{4} = 6\left(\frac{-1+9}{4}\right) = (6)(2) = \boxed{12}.$$

Exercise 5.6.18 (continued 2)

Solution (continued). We now work this problem again, but this time we use differentials to represent the substitution. This process is justified by Theorem 5.7 (Substitution in Definite Integrals) and just involves a simplified notation. We have:

$$\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right)\,d\theta = \int_{\theta=\pi}^{\theta=3\pi/2} \frac{\sec^2(\theta/6)}{\tan^5(\theta/6)}\,d\theta$$

$= \; 6\int_{u=1/\sqrt{3}}^{u=1} \frac{1}{u^5}\,du$ where $u = \tan(\theta/6)$ and so $du = \sec^2(\theta/6)/6\,d\theta$ or $6\,du = \sec^2(\theta/6)\,d\theta$; when $\theta = \pi$ then $u = g(\pi) = \tan(\pi/6) = 1/\sqrt{3}$, and when $\theta = 3\pi/2, u = g(3\pi/2) = \tan((3\pi/2)/6) = \tan(\pi/4) = 1$

$$= \; 6\int_{u=1/\sqrt{3}}^{u=1} u^{-5}\,du = 6\frac{u^{-4}}{-4}\Bigg|_{1/\sqrt{3}}^{1} = 6\frac{(1)^{-4}}{-4} - 6\frac{(1/\sqrt{3})^{-4}}{-4} = \boxed{12},$$

as above. □

Theorem 5.8

Theorem 5.8. Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof. Notice that by the Additivity property of the integral (Theorem 5.2(5)), $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$. $(*)$

(a) For f an even function, $f(-x) = f(x)$ so that

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)\,(-du) \text{ where } u = -x \text{ and so } du = -dx$$

or $-du = dx$ and when $x = -a$ then $u = -(-a) = a$, and when $x = 0$ then $u = -(0) = 0$

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Theorem 5.8 (continued 1)

Proof (continued). ...

$$\begin{aligned}\int_{-a}^{0} f(x)\,dx &= \int_{a}^{0} f(-u)\,(-du) = -\int_{a}^{0} f(u)\,du \\ &= \int_{0}^{a} f(u)\,du \text{ by Order of Integration, Theorem 5.2(1)} \\ &= \int_{0}^{a} f(x)\,dx.\end{aligned}$$

So by $(*)$,

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

as claimed.

Theorem 5.8 (continued 2)

Proof (continued). (b) For f an odd function, $f(-x) = -f(x)$ so that

$$\begin{aligned}\int_{-a}^{0} f(x)\,dx &= \int_{a}^{0} f(-u)\,(-du) \text{ where } u = -x \text{ and so } du = -dx \\ &\quad \text{or } -du = dx \text{ and when } x = -a \text{ then } u = -(-a) = a, \\ &\quad \text{and when } x = 0 \text{ then } u = -(0) = 0 \\ &= -\int_{a}^{0} (-f(u))\,du = \int_{a}^{0} f(u)\,du \\ &= -\int_{0}^{a} f(u)\,du \text{ by Order of Integration, Theorem 5.2(1)} \\ &= -\int_{0}^{a} f(x)\,dx \text{ replacing the variable of integration.}\end{aligned}$$

So by $(*)$, $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0$, as claimed. □

Exercise 5.6.14

Exercise 5.6.14. (a) Evaluate $\displaystyle\int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$. **(b)** Evaluate $\displaystyle\int_{-\pi/2}^{\pi/2}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$.

Solution. (a) We have $\displaystyle\int_{-\pi/2}^{0}\left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\,dt$

$$= \int_{-1}^{0}(2+u)(2\,du) \text{ where } u=\tan\frac{t}{2} \text{ and so } du=\frac{1}{2}\sec^2\frac{t}{2}\,dt$$

$$\text{or } 2\,du=\sec^2\frac{t}{2}\,dt \text{ and when } t=-\pi/2 \text{ then } u=\tan\frac{-\pi/2}{2}=-1,$$

$$\text{and when } t=0 \text{ then } u=\tan\frac{0}{2}=0$$

$$= \int_{-1}^{0}(4+2u)\,du=(4u+u^2)\Big|_{-1}^{0}$$

$$= (4(0)+(0)^2)-(4(-1)+(-1)^2)=\boxed{3}.$$

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$$= \int_{-1}^{0}(4+2u)\,du=(4u+u^2)\Big|_{-1}^{0}$$

$$= (4(0)+(0)^2)-(4(-1)+(-1)^2)=\boxed{3}.$$

Exercise 5.6.14 (continued)

Solution (continued). (b) We have

$$\int_{-\pi/2}^{\pi/2} \left(2 + \tan \frac{t}{2}\right) \sec^2 \frac{t}{2}\, dt$$

$$= \int_{-1}^{1} (2+u)(2\,du) \text{ where } u = \tan\frac{t}{2} \text{ and so } du = \frac{1}{2}\sec^2\frac{t}{2}\,dt$$

$$\text{or } 2\,du = \sec^2\frac{t}{2}\,dt \text{ and when } t = -\pi/2 \text{ then } u = \tan\frac{-\pi/2}{2} = -1,$$

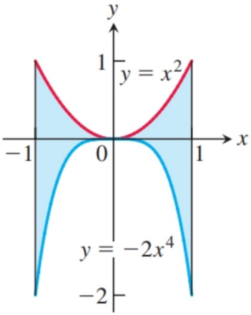
$$\text{and when } t = \pi/2 \text{ then } u = \tan\frac{\pi/2}{2} = 1$$

$$= \int_{-1}^{1} (4+2u)\,du = (4u+u^2)\Big|_{-1}^{1}$$

$$= (4(1)+(1)^2) - (4(-1)+(-1)^2) = \boxed{8}. \qquad \square$$

Exercise 5.6.58

Exercise 5.6.58. Find the area:



Solution. For $f(x) = x^2$ and $g(x) = -2x^4$ we have $f(x) \geq g(x)$ for $x \in [-1, 1]$, so by definition we have that the area is

$$A = \int_a^b (f(x) - g(x))\, dx = \int_{-1}^1 ((x^2) - (-2x^4))\, dx = \int_{-1}^1 (x^2 + 2x^4)\, dx$$

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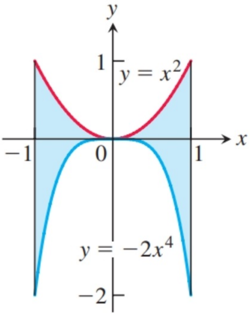

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Exercise 5.6.58 (continued)

Solution (continued). . . .

$$A = \int_a^b (f(x) - g(x))\,dx = \int_{-1}^{1} (x^2) - (-2x^4))\,dx = \int_{-1}^{1} (x^2 + 2x^4)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{2x^5}{5}\right)\Bigg|_{-1}^{1} = \left(\frac{(1)^3}{3} + \frac{2(1)^5}{5}\right) - \left(\frac{(-1)^3}{3} + \frac{2(-1)^5}{5}\right)$$

$$= \left(\frac{1}{3} + \frac{2}{5}\right) - \left(\frac{-1}{3} - \frac{2}{5}\right) = \left(\frac{5+6}{15}\right) - \left(\frac{-5-6}{15}\right) = \boxed{\frac{22}{15}}. \qquad \square$$

Example 5.6.6

Example 5.6.6. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution. We consider the graph as given in Figure 5.30:

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Solution. We consider the graph as given in Figure 5.30:

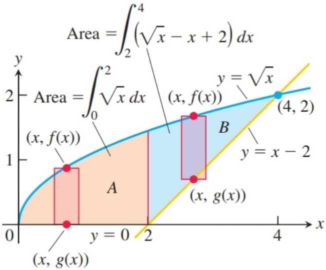

Figure 5.30

Notice that for $x \in [0, 2]$ the region is bounded above by $y = \sqrt{x}$ and below by $y = 0$. For $x \in [2, 4]$ the region is bounded above by $y = \sqrt{x}$ and below by $y = x - 2$.

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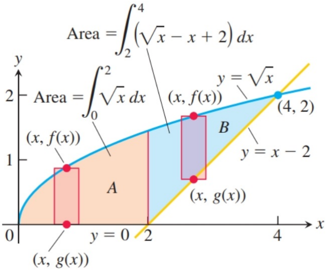

Figure 5.30

Notice that for $x \in [0, 2]$ the region is bounded above by $y = \sqrt{x}$ and below by $y = 0$. For $x \in [2, 4]$ the region is bounded above by $y = \sqrt{x}$ and below by $y = x - 2$.

Example 5.6.6 (continued)

Solution (continued). So we can express the area as the sum of two integrals:

$$A = \int_0^2 (\sqrt{x} - 0)\, dx + \int_2^4 (\sqrt{x} - (x-2))\, dx = \int_0^2 x^{1/2}\, dx + \int_2^4 (x^{1/2} - x + 2)\, dx$$

$$= \left. \left(\frac{2}{3} x^{3/2} \right) \right|_0^2 + \left. \left(\frac{2}{3} x^{3/2} - \frac{1}{2} x^2 + 2x \right) \right|_2^4$$

$$= \left(\frac{2}{3}(2)^{3/2} - \frac{2}{3}(0)^{3/2} \right) + \left(\frac{2}{3}(4)^{3/2} - \frac{1}{2}(4)^2 + 2(4) \right)$$

$$- \left(\frac{2}{3}(2)^{3/2} - \frac{1}{2}(2)^2 + 2(2) \right) = \frac{2}{3}(2)^{3/2} + \frac{2}{3}(4)^{3/2} - 8 + 8 - \frac{2}{3}(2)^{3/2} + 2 - 4$$

$$= \frac{2}{3}(8) - 2 = \boxed{\frac{10}{3}}. \qquad \square$$

Exercise 5.6.62

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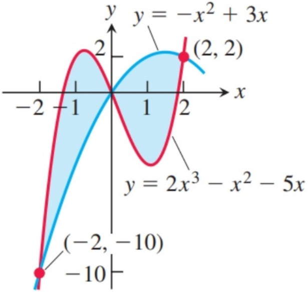

Solution. From the graph we see that $f(x) = 2x^3 - x^2 - 5x \geq -x^2 + 3x = g(x)$ for $x \in [-2, 0]$, and $g(x) = -x^2 + 3x \geq 2x^3 - x^2 - 5x = f(x)$ for $x \in [0, 2]$. So the area can be found, by definition, by adding the integral of $f - g$ over the interval $[-2, 0]$ to the integral of $g - f$ over the interval $[0, 2]$...

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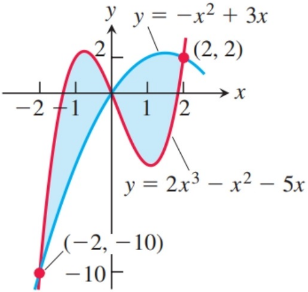

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Exercise 5.6.62 (continued)

Solution (continued). ...

$$A = \int_{-2}^{0} ((2x^3 - x^2 - 5x) - (-x^2 + 3x))\,dx + \int_{0}^{2} ((-x^2 + 3x) - (2x^3 - x^2 - 5x))\,dx$$

$$= \int_{-2}^{0} (2x^3 - 8x)\,dx + \int_{0}^{2} (-2x^3 + 8x)\,dx$$

$$= \left(\frac{2x^4}{4} - \frac{8x^2}{2}\right)\Big|_{-2}^{0} + \left(\frac{8x^2}{2} - \frac{2x^4}{4}\right)\Big|_{0}^{2} = \left(\frac{x^4}{2} - 4x^2\right)\Big|_{-2}^{0} + \left(4x^2 - \frac{x^4}{2}\right)\Big|_{0}^{2}$$

$$= (0 - 0) - \left(\frac{(-2)^4}{2} - 4(-2)^2\right) + \left(4(2)^2 - \frac{(2)^4}{2}\right) - (0 - 0)$$

$$= -(8 - 16) + (16 - 8) = 8 + 8 = \boxed{16}. \qquad \square$$

Exercise 5.6.90

Exercise 5.6.90. Find the area of the region enclosed by the line $y = x$ and the curve $y = \sin(\pi x/2)$.

Solution. Notice that the amplitude of $y = \sin(\pi x/2)$ is 1 and the period is $2\pi/(\pi/2) = 4$. From the graph we see that $y = \sin(\pi x/2)$ and $y = x$ intersect at $(-1, -1)$, $(0, 0)$, and $(1, 1)$:

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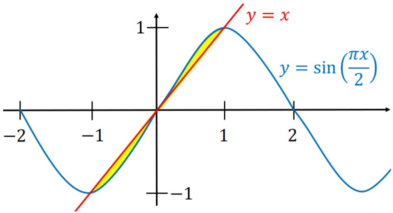

We have $x \geq \sin(\pi x/2)$ for $x \in [-1, 0]$, and $\sin(\pi x/2) \geq x$ for $x \in [0, 1]$.

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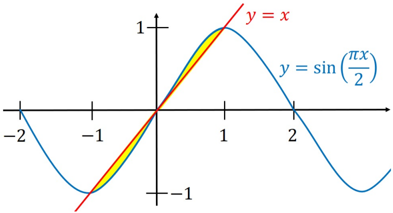

We have $x \geq \sin(\pi x/2)$ for $x \in [-1, 0]$, and $\sin(\pi x/2) \geq x$ for $x \in [0, 1]$.

Exercise 5.6.90 (continued)

Solution (continued). . . . We have $x \geq \sin(\pi x/2)$ for $x \in [-1, 0]$, and $\sin(\pi x/2) \geq x$ for $x \in [0, 1]$. So the area enclosed by $y = x$ and $y = \sin(\pi x/2)$ is given by the sum of the integrals:

$$A = \int_{-1}^{0} (x - \sin(\pi x/2))\, dx + \int_{0}^{1} (\sin(\pi x/2) - x)\, dx$$

$$= \left(\frac{x^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right)\Bigg|_{-1}^{0} + \left(-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right)\Bigg|_{0}^{1}$$

$$= \left(\left(\frac{(0)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi(0)}{2}\right)\right) - \left(\frac{(-1)^2}{2} + \frac{2}{\pi}\cos\left(\frac{\pi(-1)}{2}\right)\right)\right)$$

$$+ \left(\left(-\frac{2}{\pi}\cos\left(\frac{\pi(1)}{2}\right) - \frac{(1)^2}{2}\right) - \left(-\frac{2}{\pi}\cos\left(\frac{\pi(0)}{2}\right) - \frac{(0)^2}{2}\right)\right)$$

$$= \left(0 + \frac{2}{\pi}(1)\right) - \left(\frac{1}{2} + \frac{2}{\pi}(0)\right) + \left(-\frac{2}{\pi}(0) - \frac{1}{2}\right) - \left(-\frac{2}{\pi}(1) - 0\right) = \boxed{2\left(\frac{2}{\pi} - \frac{1}{2}\right)}.$$

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$$= \left(0 + \frac{2}{\pi}(1)\right) - \left(\frac{1}{2} + \frac{2}{\pi}(0)\right) + \left(-\frac{2}{\pi}(0) - \frac{1}{2}\right) - \left(-\frac{2}{\pi}(1) - 0\right) = \boxed{2\left(\frac{2}{\pi} - \frac{1}{2}\right)}.$$

Exercise 5.6.78

Exercise 5.6.78. Find the area of the region bounded by the curves $x - y^2 = 0$ and $x + 2y^2 = 3$.

Solution. We need to find where these curves intersect. We have $x = y^2$ and $x = 3 - 2y^2$, so to find the intersection we set the x-coordinates equal and consider $y^2 = 3 - 2y^2$ or $3y^2 = 3$ or $y = \pm 1$. Notice that for both curves we have x as a function of y. The graphs are:

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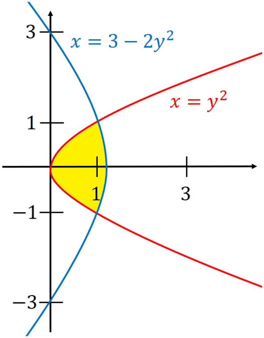

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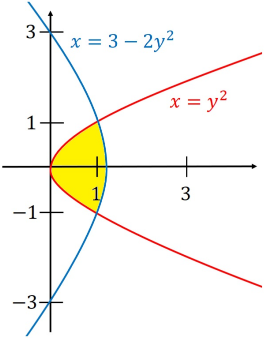
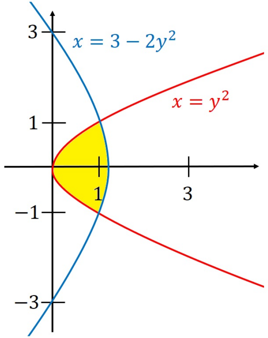


Exercise 5.6.78 (continued)

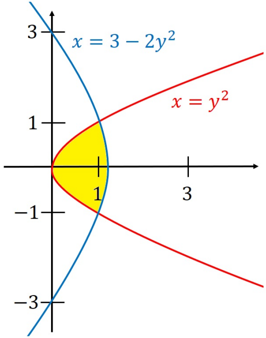
Solution (continued).



Notice that $x = 3 - 2y^2$ is on the right and $x = y^2$ is on the left. So we integrate with respect to y from -1 to 1 the difference $(3 - 2y^2) - (y^2)$:

Exercise 5.6.78 (continued)

Solution (continued).



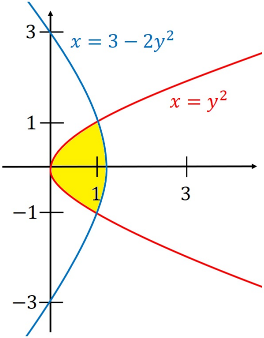
Notice that $x = 3 - 2y^2$ is on the right and $x = y^2$ is on the left. So we integrate with respect to y from -1 to 1 the difference $(3 - 2y^2) - (y^2)$:

$$\int_{-1}^{1} (3 - 2y^2) - (y^2)\, dy = \int_{-1}^{1} 3 - 3y^2\, dy = (3y - y^3)\Big|_{-1}^{1}$$

$$= (3(1) - (1)^3) - (3(-1) - (-1)^3) = \boxed{4}. \qquad \square$$

Exercise 5.6.78 (continued)

Solution (continued).



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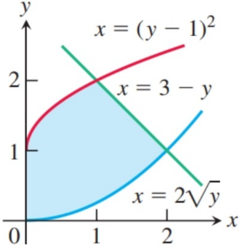
Exercise 5.6.108

Exercise 5.6.108. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right be the line $x = 3 - y$:



Solution. Notice for $y \in [0,1]$ that the graph of $x = 2\sqrt{y}$ is on the right of the region and $x = 0$ is on the left. For $y \in [1,2]$ the graph of $x = 3 - y$ is on the right of the region and $x = (y-1)^2$ is on the left. So the area is the sum of two integrals with respect to y: ...

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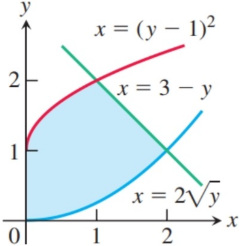

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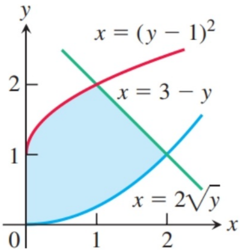
Exercise 5.6.108 (continued 1)

Solution (continued).



So the area is the sum of two integrals with respect to y:

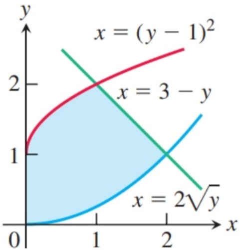
$$A = \int_0^1 (2\sqrt{y} - 0)\, dy + \int_1^2 ((3 - y) - (y - 1)^2)\, dy$$

$$= \int_0^1 2y^{1/2}\, dy + \int_1^2 -y^2 + y + 2\, dy$$

$$= 2\left(\frac{2}{3}y^{3/2}\right)\Big|_0^1 + \left(\frac{-1}{3}y^3 + \frac{1}{2}y^2 + 2y\right)\Big|_1^2$$

Exercise 5.6.108 (continued 2)

Solution (continued).



$$A = 2\left(\frac{2}{3}y^{3/2}\right)\Bigg|_0^1 + \left(\frac{-1}{3}y^3 + \frac{1}{2}y^2 + 2y\right)\Bigg|_1^2$$

$$= \left(\frac{4}{3}(1)^{3/2} - \frac{4}{3}(0)^{3/2}\right) + \left(\frac{-(2)^3}{3} + \frac{(2)^2}{2} + 2(2)\right) - \left(\frac{-(1)^3}{3} + \frac{(1)^2}{2} + 2(1)\right)$$

$$= \frac{4}{3} + \left(\frac{-8}{3} + 2 + 4\right) - \left(\frac{-1}{3} + \frac{1}{2} + 2\right) = 3 - \frac{1}{2} = \boxed{\frac{5}{2}}. \quad \square$$

Exercise 5.6.114

Exercise 5.6.114. Show that if f is continuous, then $\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$.

Solution. We have

$$\begin{aligned}
\int_0^1 f(1-x)\,dx &= \int_1^0 f(u)\,(-du) \text{ where } u = 1-x \text{ and so } du = -dx \\
&\quad \text{or } -du = dx \text{ and when } x = 0 \text{ then } u = 1-(0) = 1, \\
&\quad \text{and when } x = 1 \text{ then } u = 1-(1) = 0 \\
&= -\int_1^0 f(u)\,du = \int_0^1 f(u)\,du \text{ by Order of Integration,} \\
&\quad \text{Theorem 5.2(1)} \\
&= \int_0^1 f(x)\,dx \text{ replacing the variable of integration.}
\end{aligned}$$

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Exercise 5.6.118

Exercise 5.6.118. By using a substitution, prove that for all positive numbers x and y, $\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt$.

Proof. We have

$$\begin{aligned}
\int_x^{xy} \frac{1}{t}\,dt &= \int_1^y \frac{1}{ux} x\,du \text{ where } u = t/x \text{ (or } t = ux) \text{ and so } du = 1/x\,dt \\
&\quad \text{or } x\,du = dt \text{ and when } t = x \text{ then } u = (x)/x = 1, \\
&\quad \text{and when } t = xy \text{ then } u = (xy)/x = y \\
&= \int_1^y \frac{1}{u}\,du \\
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