Chapter 1. Functions
1.1. Functions and Their Graphs—Examples and Proofs
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Exercise 1.1.4

Exercise 1.1.4. Find the domain and range of \( g(x) = \sqrt{x^2 - 3x} \).

Solution. We cannot take square roots of negative numbers, so we need \( x^2 - 3x \geq 0 \) or \( x(x - 3) \geq 0 \). Now \( x^2 - 3x = x(x - 3) = 0 \) for \( x = 0 \) and \( x = 3 \). Since the graph of \( y = x^2 - 3x \) is a parabola, then it represents a continuous function, and the only way it can change sign (from positive to negative, or from negative to positive) is by passing through the value 0. So if \( x^2 - 3x = x(x - 3) \) changes sign, then it does it at \( x = 0 \) or \( x = 3 \) and so \( x^2 - 3x = x(x - 3) \) has the same sign on the intervals \((-\infty, 0)\), \((0, 3)\), and \((3, \infty)\). We just need to test the sign of \( x^2 - 3x \) at a test value from each interval.
Exercise 1.1.4. Find the domain and range of \( g(x) = \sqrt{x^2 - 3x} \).

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<table>
<thead>
<tr>
<th>interval</th>
<th>test value ( k )</th>
<th>( f(k) = k^2 - 3k )</th>
<th>Sign of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((−∞, 0))</td>
<td>−1</td>
<td>((-1)^2 - 3(-1) = 4)</td>
<td>+</td>
</tr>
<tr>
<td>((0, 3))</td>
<td>1</td>
<td>((1)^2 - 3(1) = -2)</td>
<td>−</td>
</tr>
<tr>
<td>((3, ∞))</td>
<td>4</td>
<td>((4)^2 - 3(4) = 4)</td>
<td>+</td>
</tr>
</tbody>
</table>
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<tr>
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<td>$-1$</td>
<td>$(-1)^2 - 3(-1) = 4$</td>
<td>$+$</td>
</tr>
<tr>
<td>$(0, 3)$</td>
<td>$1$</td>
<td>$(1)^2 - 3(1) = -2$</td>
<td>$-$</td>
</tr>
<tr>
<td>$(3, \infty)$</td>
<td>$4$</td>
<td>$(4)^2 - 3(4) = 4$</td>
<td>$+$</td>
</tr>
</tbody>
</table>
Exercise 1.1.4 (continued)

Solution (continued). We see that $x^2 - 3x > 0$ for $x \in (-\infty, 0) \cup (3, \infty)$, and so $x^2 - 3x \geq 0$ for $x \in (-\infty, 0] \cup [3, \infty)$. That is, the domain of $g$ is $(-\infty, 0] \cup [3, \infty)$.

For $r$ in the range of $g$ (that is, if $r = \sqrt{x^2 - 3x}$ for some $x$ in the domain of $g$), we must have $r \geq 0$ (since square roots are never negative; see Appendix A1. Real Numbers and the Real Line). For such $r$, if $r = g(x) = \sqrt{x^2 - 3x}$ then $r^2 = (\sqrt{x^2 - 3x})^2 = x^2 - 3x$ or $x^2 - 3x - r^2 = 0$ and by the quadratic equation,

$$x = \frac{-(3) \pm \sqrt{(-3)^2 - 4(1)(-r^2)}}{2(1)} = \frac{3 \pm \sqrt{9 + 4r^2}}{2}.$$ 

Since $9 + 4r^2 \geq 0$ then such an $x$ exists and so $r$ is in the range of $g$, provided $r \geq 0$. That is, the range of $g$ is $[0, \infty)$. □
Exercise 1.1.4 (continued)

**Solution (continued).** We see that \( x^2 - 3x > 0 \) for \( x \in (-\infty, 0) \cup (3, \infty) \), and so \( x^2 - 3x \geq 0 \) for \( x \in (-\infty, 0] \cup [3, \infty) \). That is, the domain of \( g \) is \((-\infty, 0] \cup [3, \infty)\).

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Notice that \( \frac{3 - \sqrt{9 + 4r^2}}{2} \leq 0 \) and \( \frac{3 + \sqrt{9 + 4r^2}}{2} \geq 3 \). So there are two \( x \) values that produce output value \( r \geq 0 \) (one in \((-\infty, 0)\) and one in \((3, \infty))\).
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Solution (continued). We see that $x^2 - 3x > 0$ for $x \in (-\infty, 0) \cup (3, \infty)$, and so $x^2 - 3x \geq 0$ for $x \in (-\infty, 0] \cup [3, \infty)$. That is, the domain of $g$ is $(-\infty, 0] \cup [3, \infty)$.

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Since $9 + 4r^2 \geq 0$ then such an $x$ exists and so $r$ is in the range of $g$, provided $r \geq 0$. That is, the range of $g$ is $[0, \infty)$. □

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Exercise 1.1.26

Exercise 1.1.26. Graph the function:

\[ g(x) = \begin{cases} 
1 - x, & 0 \leq x \leq 1 \\
2 - x, & 1 < x \leq 2.
\end{cases} \]

Solution. The graph of \( y = 1 - x \) is a line of slope \( m_1 = -1 \) containing the point \((0, 1)\) (this is the \( y \)-intercept). The graph of \( y = 2 - x \) is a line of slope \( m_2 = -1 \) containing the point \((2, 0)\) (this is the \( x \)-intercept). So we graph \( y = 1 - x \) for \( 0 \leq x \leq 1 \) and \( y = 2 - x \) for \( 1 < x \leq 2 \):
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Exercise 1.1.58. Determine whether the function $h(t) = 2|t| + 1$ is even, odd, or neither.

Solution. We replace the independent variable $t$ with $-t$ to test for evenness or oddness:

$$h(-t) = 2|(-t)| + 1 = 2|-1||t| + 1 = 2(1)|t| + 1 = 2|t| + 1 = h(t).$$

So we have $h(-t) = h(t)$ and hence $h$ is an even function. □
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So we have \( h(-t) = h(t) \) and hence \( h \) is an even function. □
Exercise 1.1.68. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long. (a) Express the y-coordinate of $P$ in terms of $x$. (b) Express the area of the rectangle in terms of $x$.

**Proof.** (a) Let $O$ represent the origin. The $y$-axis bisects the right angle in the given isosceles triangle, so triangle $AOB$ is similar to the given isosceles triangle and therefore is itself an isosceles triangle. That is, the length of segment $OB$ is 1 and so point $B$ is $(0, 1)$.
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(a) Express the $y$-coordinate of $P$ in terms of $x$.

**Proof (continued).** So the equation of the line passing through $A = (x_1, y_1) = (1, 0)$ and $B = (x_2, y_2) = (0, 1)$ has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(1) - (0)}{(0) - (1)} = -1$$

and hence is, by the point slope formula,

$$y - y_1 = m(x - x_1)$$

or

$$y - (0) = (-1)(x - 1)$$

or

$$y = -x + 1.$$ 

So point $P$ is of the form $(x, y) = (x, -x + 1)$. That is, the $y$-coordinate of $P$ in terms of $x$ is $-x + 1$. 


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y - y_1 = m(x - x_1) \quad \text{or} \quad y - (0) = (-1)(x - 1) \quad \text{or} \quad y = -x + 1.
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Proof (continued). (b) Since point $P$ is of the form $(x, y) = (x, -x + 1)$, the width of the rectangle is $2x$, and the height of the rectangle is $y$, then the area of the rectangle is $A = 2xy$ or $A = 2x(-x + 1)$. □

Notice that the values given in (a) and (b) are only (physically) meaningful for $x \in [0, 1]$. 
(b) Express the area of the rectangle in terms of $x$.

Proof (continued). (b) Since point $P$ is of the form $(x, y) = (x, -x + 1)$, the width of the rectangle is $2x$, and the height of the rectangle is $y$, then the area of the rectangle is $A = 2xy$ or $A = 2x(-x + 1)$. □

Notice that the values given in (a) and (b) are only (physically) meaningful for $x \in [0, 1]$. 
Exercise 1.1.76. Industrial Costs.

A power plant sits next to a river where the river is 800 ft wide. To lay a new cable from the plant to a location in the city 2 mi downstream on the opposite side costs $180 per foot across the river and $100 per foot along the land.

(a) Suppose that the cable goes from the plant to a point $Q$ on the opposite side that is $x$ ft from the point $P$ directly opposite the plant. Write a function $C(x)$ that gives the cost of laying the cable in terms of the distance $x$.

(b) Generate a table of values to determine if the least expensive location for point $Q$ is less than 2000 ft or greater than 2000 ft from point $P$.

Solution. (a) First, 2 mile = (2 mile)(5,280 ft/mile) = 10,560 ft. We see that there is $(10,560 - x)$ ft of cable along the land.
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Solution. (a) First, 2 mile = (2 mile)(5,280 ft/mile) = 10,560 ft. We see that there is $(10,560 - x)$ ft of cable along the land.
Solution (continued). Since the Power Plant, point $P$, and point $Q$ form a right triangle, then by the Pythagorean Theorem we see that the amount of cable across the river is $\sqrt{(800)^2 + x^2}$ ft. Since the cable costs $180$ per foot across the river and there is $\sqrt{(800)^2 + x^2}$ ft of cable across the river, then the cost of this part of the cable is $180\sqrt{(800)^2 + x^2}$. Since the cable costs $100$ per foot along the land and there is $(10,560 - x)$ ft of cable along the land, then the cost of this part of the cable is $100(10,560 - x)$. So the total cost of the cable is

$$C(x) = 180\sqrt{(800)^2 + x^2} + 100(10,560 - x) \text{ dollars}.$$
Exercise 1.1.76 (continued 2)

(b) Generate a table of values to determine if the least expensive location for point $Q$ is less than 2000 ft or greater than 2000 ft from point $P$.

Solution (continued). We make a table of values of $C(x) = 180\sqrt{(800)^2 + x^2} + 100(10,560 - x)$ dollars:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$C(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2300</td>
<td>$1,264,328.64$</td>
</tr>
<tr>
<td>2200</td>
<td>$1,257,369.20$</td>
</tr>
<tr>
<td>2100</td>
<td>$1,250,499.69$</td>
</tr>
<tr>
<td>2000</td>
<td>$1,243,731.87$</td>
</tr>
<tr>
<td>1900</td>
<td>$1,237,079.51$</td>
</tr>
<tr>
<td>1800</td>
<td>$1,230,558.88$</td>
</tr>
<tr>
<td>1700</td>
<td>$1,224,289.30$</td>
</tr>
</tbody>
</table>

It appears that the least expensive location for point $Q$ is less than 2000 ft from point $P$. □