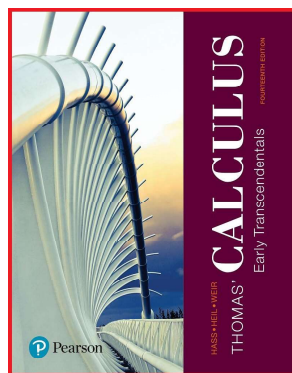


# Calculus 1

## Chapter 1. Functions

### 1.3. Trigonometric Functions—Examples and Proofs



#### Exercise 1.3.6

### Exercise 1.3.6

**Exercise 1.3.6.** Finish the following table of trigonometric values of some special angles:

| $\theta$      | $-3\pi/2$ | $-\pi/3$ | $-\pi/6$ | $\pi/4$ | $5\pi/6$ |
|---------------|-----------|----------|----------|---------|----------|
| $\sin \theta$ |           |          |          |         |          |
| $\cos \theta$ |           |          |          |         |          |
| $\tan \theta$ |           |          |          |         |          |
| $\cot \theta$ |           |          |          |         |          |
| $\sec \theta$ |           |          |          |         |          |
| $\csc \theta$ |           |          |          |         |          |

#### Exercise 1.3.2

### Exercise 1.3.2

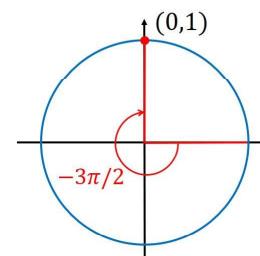
**Exercise 1.3.2.** A central angle in a circle of radius 8 is subtended by an arc of length  $10\pi$ . Find the angle's radian and degree measure.

**Solution.** The radius is  $r = 8$  and the arc length is  $s = 10\pi$ . Since  $\theta = s/r$ , then here  $\theta = (10\pi)/8 = \boxed{5\pi/4}$ .

To convert  $\theta$  to degrees, we multiply by the conversion factor of  $180^\circ/\pi$  (or, if you like,  $(180/\pi)^\circ/\text{radian}$ ; but remember that that radians are unitless). So we have  $\theta = (5\pi/4)(180^\circ/\pi) = \boxed{225^\circ}$ .  $\square$

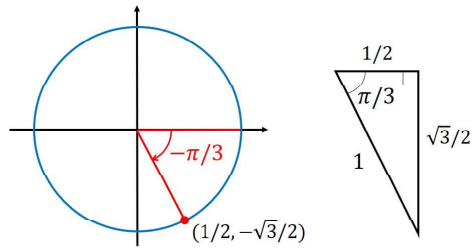
#### Exercise 1.3.6

### Exercise 1.3.6 (continued 1)



**Solution.** For  $\theta = -3\pi/2$ , the point on the unit circle and terminal side of  $\theta$  is  $(x, y) = (0, 1)$ . By definition, since  $r = 1$  on the unit circle, we have  $\sin(-3\pi/2) = y/r = 1/1 = 1$ ,  $\cos(-3\pi/2) = x/r = 0/1 = 0$ ,  $\sec(-3\pi/2) = r/x$  is undefined,  $\csc(-3\pi/2) = r/y = 1/1 = 1$ ,  $\tan(-3\pi/2) = y/x$  is undefined, and  $\cot(-3\pi/2) = x/y = 0/1 = 0$ .

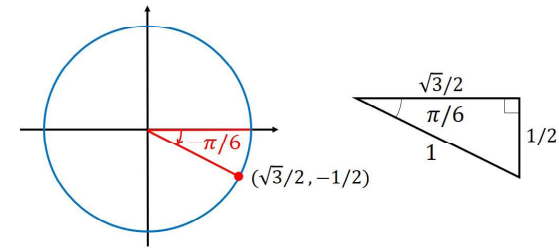
## Exercise 1.3.6 (continued 2)



**Solution (continued).** For  $\theta = -\pi/3$ , we use the special right triangle containing an angle of  $\pi/3$  to find that the point on the unit circle and terminal side of  $\theta$  is  $(x, y) = (1/2, -\sqrt{3}/2)$ . By definition, since  $r = 1$  on the unit circle, we have  $\sin(-\pi/3) = y/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2$ ,  $\cos(-\pi/3) = x/r = (1/2)/(1) = 1/2$ ,  $\sec(-\pi/3) = r/x = (1)/(1/2) = 2$ ,  $\csc(-\pi/3) = r/y = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}$ ,  $\tan(-\pi/3) = y/x = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$ , and  $\cot(-\pi/3) = x/y = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}$ .

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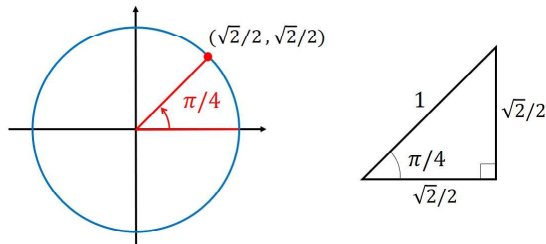
## Exercise 1.3.6 (continued 3)



**Solution (continued).** For  $\theta = -\pi/6$ , we use the special right triangle containing an angle of  $\pi/6$  to find that the point on the unit circle and terminal side of  $\theta$  is  $(x, y) = (\sqrt{3}/2, -1/2)$ . By definition, since  $r = 1$  on the unit circle, we have  $\sin(-\pi/6) = y/r = (-1/2)/(1) = -1/2$ ,  $\cos(-\pi/6) = x/r = (\sqrt{3}/2)/(1) = \sqrt{3}/2$ ,  $\sec(-\pi/6) = r/x = (1)/(\sqrt{3}/2) = 2/\sqrt{3}$ ,  $\csc(-\pi/6) = r/y = (1)/(-1/2) = -2$ ,  $\tan(-\pi/6) = y/x = (-1/2)/(\sqrt{3}/2) = -1/\sqrt{3}$ , and  $\cot(-\pi/6) = x/y = (\sqrt{3}/2)/(-1/2) = -\sqrt{3}$ .

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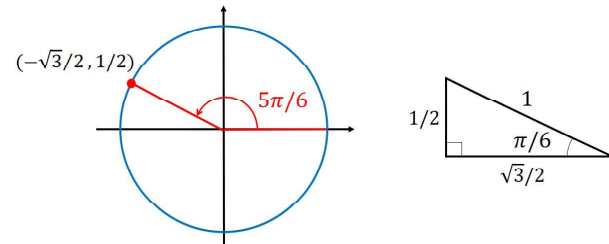
## Exercise 1.3.6 (continued 4)



**Solution (continued).** For  $\theta = \pi/4$ , we use the special right triangle containing an angle of  $\pi/4$  to find that the point on the unit circle and terminal side of  $\theta$  is  $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$ . By definition, since  $r = 1$  on the unit circle, we have  $\sin(\pi/4) = y/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$ ,  $\cos(\pi/4) = x/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$ ,  $\sec(\pi/4) = r/x = (1)/(\sqrt{2}/2) = \sqrt{2}$ ,  $\csc(\pi/4) = r/y = (1)/(\sqrt{2}/2) = \sqrt{2}$ ,  $\tan(\pi/4) = y/x = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$ , and  $\cot(\pi/4) = x/y = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$ .

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## Exercise 1.3.6 (continued 5)



**Solution (continued).** For  $\theta = 5\pi/6$ , we use the special right triangle containing an angle of  $5\pi/6$  to find that the point on the unit circle and terminal side of  $\theta$  is  $(x, y) = (-\sqrt{3}/2, 1/2)$ . By definition, since  $r = 1$  on the unit circle, we have  $\sin(5\pi/6) = y/r = (1/2)/(1) = 1/2$ ,  $\cos(5\pi/6) = x/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2$ ,  $\sec(5\pi/6) = r/x = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}$ ,  $\csc(5\pi/6) = r/y = (1)/(1/2) = 2$ ,  $\tan(5\pi/6) = y/x = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}$ , and  $\cot(5\pi/6) = x/y = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$ .

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## Exercise 1.3.6 (continued 6)

**Solution (continued).** We therefore have:

| $\theta$      | $-3\pi/2$ | $-\pi/3$      | $-\pi/6$      | $\pi/4$      | $5\pi/6$      |
|---------------|-----------|---------------|---------------|--------------|---------------|
| $\sin \theta$ | 1         | $-\sqrt{3}/2$ | $-1/2$        | $\sqrt{2}/2$ | $1/2$         |
| $\cos \theta$ | 0         | $1/2$         | $\sqrt{3}/2$  | $\sqrt{2}/2$ | $-\sqrt{3}/2$ |
| $\tan \theta$ | UND       | $-\sqrt{3}$   | $-1/\sqrt{3}$ | 1            | $-1/\sqrt{3}$ |
| $\cot \theta$ | 0         | $-1/\sqrt{3}$ | $-\sqrt{3}$   | 1            | $-\sqrt{3}$   |
| $\sec \theta$ | UND       | 2             | $2/\sqrt{3}$  | $\sqrt{2}$   | $-2/\sqrt{3}$ |
| $\csc \theta$ | 1         | $-2/\sqrt{3}$ | -2            | $\sqrt{2}$   | 2             |

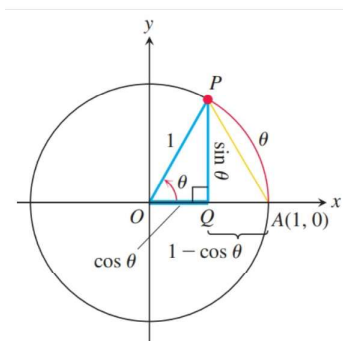
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## Example 1.3.A

## Example 1.3.A

**Example 1.3.A.** For any angle  $\theta$  measured in radians, we have  $-\theta \leq \sin \theta \leq \theta$  and  $-\theta \leq 1 - \cos \theta \leq \theta$ .



**Solution.** As in Figure 1.47, we put  $\theta$  in standard position. Since the circle is a unit circle (that is,  $r = 1$ ), then  $|\theta|$  equals the length of the circular arc  $AP$ .

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## Exercise 1.3.31

**Exercise 1.3.31.** Use the addition formulas to derive the identity  $\cos\left(x - \frac{\pi}{2}\right) = \sin x$ .

**Solution.** We have the formula  $\cos(A - B) = \cos A \cos B + \sin A \sin B$ , so with  $A = x$  and  $B = \pi/2$  we have

$$\cos(x - \pi/2) = \cos x \cos \pi/2 + \sin x \sin \pi/2 = \cos x(0) + \sin x(1) = \sin x.$$

□

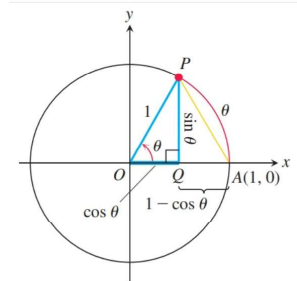
Notice that  $x$  and  $x - \pi/2$  are complementary angles since  $(x) + (x - \pi/2) = \pi/2$ . So this exercise shows that the sine of an angle equals the cosine of its complement; *this* is why cosine is called “cosine.”

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## Example 1.3.A

## Example 1.3.A (continued)

**Solution (continued).** We see from the figure that the length of line segment  $AP$  is less than or equal to  $|\theta|$ . Triangle  $APQ$  is a right triangle with sides of length  $QP = |\sin \theta|$  and  $AQ = 1 - \cos \theta$ . So by the Pythagorean Theorem (and the fact that  $AP \leq |\theta|$ ) we have  $\sin^2 \theta + (1 - \cos \theta)^2 = (AP)^2 \leq \theta^2$ . So we have both  $\sin^2 \theta \leq \theta^2$  and  $(1 - \cos \theta)^2 \leq \theta^2$ . Taking square roots (and observing that the square root function is an increasing function so that it preserves inequalities),  $\sqrt{\sin^2 \theta} \leq \sqrt{\theta^2}$  and  $\sqrt{(1 - \cos \theta)^2} \leq \sqrt{\theta^2}$ , or  $|\sin \theta| \leq |\theta|$  and  $|1 - \cos \theta| \leq |\theta|$ . These two inequalities imply that  $-\theta \leq \sin \theta \leq \theta$  and  $-\theta \leq 1 - \cos \theta \leq \theta$ , as claimed (see Appendix A.1. Real Numbers and the Real Line where intervals are related to absolute values). □



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## Exercise 1.3.68

**Exercise 1.3.68.** The general sine curve is

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D.$$

For  $y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2}$  identify  $A$ ,  $B$ ,  $C$ , and  $D$  and sketch the graph.

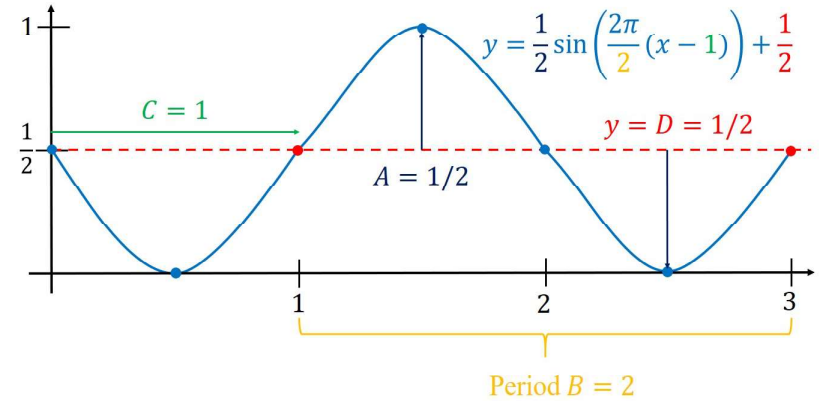
**Solution.** First we write

$$y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2} = \frac{1}{2} \sin(\pi(x - 1)) + \frac{1}{2} = \frac{1}{2} \sin\left(\frac{2\pi}{2}(x - 1)\right) + \frac{1}{2}.$$

We have  $A = 1/2$ ,  $B = 2$ ,  $C = 1$ , and  $D = 1/2$ . Now  $A$  is the amplitude,  $B$  is the period,  $C$  is the horizontal shift, and  $y = D$  is the axis. ...

## Exercise 1.3.68 (continued)

**Solution (continued).** We have  $A = 1/2$ ,  $B = 2$ ,  $C = 1$ , and  $D = 1/2$ . Now  $A$  is the amplitude,  $B$  is the period,  $C$  is the horizontal shift, and  $y = D$  is the axis.



□