Chapter 1. Functions
1.3. Trigonometric Functions—Examples and Proofs
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Exercise 1.3.2. A central angle in a circle of radius 8 is subtended by an arc of length $10\pi$. Find the angle’s radian and degree measure.

Solution. The radius is $r = 8$ and the arc length is $s = 10\pi$. Since $\theta = s/r$, then here $\theta = (10\pi)/8 = \frac{5\pi}{4}$.
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To convert $\theta$ to degrees, we multiply by the conversion factor of $180^\circ/\pi$ (or, if you like, $(180/\pi)^\circ$/radian; but remember that that radians are unitless). So we have $\theta = (5\pi/4)(180^\circ/\pi) = 225^\circ$. □
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Exercise 1.3.6. Finish the following table of trigonometric values of some special angles:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$-\frac{3\pi}{2}$</th>
<th>$-\frac{\pi}{3}$</th>
<th>$-\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td></td>
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</tr>
<tr>
<td>$\cos \theta$</td>
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<td></td>
</tr>
<tr>
<td>$\tan \theta$</td>
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<td></td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sec \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\csc \theta$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Exercise 1.3.6 (continued 1)

Solution. For $\theta = -3\pi/2$, the point on the unit circle and terminal side of $\theta$ is $(x, y) = (0, 1)$. By definition, since $r = 1$ on the unit circle, we have

\[
\begin{align*}
\sin(-3\pi/2) &= y/r = 1/1 = 1, \\
\cos(-3\pi/2) &= x/r = 0/1 = 0, \\
\sec(-3\pi/2) &= r/x \text{ is undefined}, \\
\csc(-3\pi/2) &= r/y = 1/1 = 1, \\
\tan(-3\pi/2) &= y/x \text{ is undefined}, \\
\cot(-3\pi/2) &= x/y = 0/1 = 0.
\end{align*}
\]
Solution (continued). For $\theta = -\pi/3$, we use the special right triangle containing an angle of $\pi/3$ to find that the point on the unit circle and terminal side of $\theta$ is $(x, y) = (1/2, -\sqrt{3}/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(-\pi/3) = y/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2$, $\cos(-\pi/3) = x/r = (1/2)/(1) = 1/2$, $\sec(-\pi/3) = r/x = (1)/(1/2) = 2$, $\csc(-\pi/3) = r/y = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}$, $\tan(-\pi/3) = y/x = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$, and $\cot(-\pi/3) = x/y = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}$. 
Solution (continued). For $\theta = -\pi/6$, we use the special right triangle containing an angle of $\pi/6$ to find that the point on the unit circle and terminal side of $\theta$ is $(x, y) = (\sqrt{3}/2, -1/2)$. By definition, since $r = 1$ on the unit circle, we have 

\[
\sin(-\pi/6) = y/r = (-1/2)/(1) = -1/2,
\]

\[
\cos(-\pi/6) = x/r = (\sqrt{3}/2)/(1) = \sqrt{3}/2,
\]

\[
\sec(-\pi/6) = r/x = (1)/(\sqrt{3}/2) = 2/\sqrt{3},
\]

\[
\csc(-\pi/6) = r/y = (1)/(-1/2) = -2,
\]

\[
\tan(-\pi/6) = y/x = (-1/2)/(\sqrt{3}/2) = -1/\sqrt{3}, \text{ and}
\]

\[
\cot(-\pi/6) = x/y = (\sqrt{3}/2)/(-1/2) = -\sqrt{3}.
\]
Solution (continued). For $\theta = \pi/4$, we use the special right triangle containing an angle of $\pi/4$ to find that the point on the unit circle and terminal side of $\theta$ is $(x, y) = (\sqrt{2}/2, \sqrt{2}/2)$. By definition, since $r = 1$ on the unit circle, we have $\sin(\pi/4) = y/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$,
$\cos(\pi/4) = x/r = (\sqrt{2}/2)/(1) = \sqrt{2}/2$,
$\sec(\pi/4) = r/x = (1)/(\sqrt{2}/2) = \sqrt{2}$,
$\csc(\pi/4) = r/y = (1)/(\sqrt{2}/2) = \sqrt{2}$,
$\tan(\pi/4) = y/x = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$, and
$\cot(\pi/4) = x/y = (\sqrt{2}/2)/(\sqrt{2}/2) = 1$. 
Solution (continued). For \( \theta = 5\pi/6 \), we use the special right triangle containing an angle of \( 5\pi/6 \) to find that the point on the unit circle and terminal side of \( \theta \) is \((x, y) = (-\sqrt{3}/2, 1/2)\). By definition, since \( r = 1 \) on the unit circle, we have \[ \sin(5\pi/6) = y/r = (1/2)/(1) = 1/2, \]
\[ \cos(5\pi/6) = x/r = (-\sqrt{3}/2)/(1) = -\sqrt{3}/2, \]
\[ \sec(5\pi/6) = r/x = (1)/(-\sqrt{3}/2) = -2/\sqrt{3}, \]
\[ \csc(5\pi/6) = r/y = (1)/(1/2) = 2, \]
\[ \tan(5\pi/6) = y/x = (1/2)/(-\sqrt{3}/2) = -1/\sqrt{3}, \text{ and} \]
\[ \cot(5\pi/6) = x/y = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}. \]
Solution (continued). We therefore have:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$-\frac{3\pi}{2}$</th>
<th>$-\frac{\pi}{3}$</th>
<th>$-\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin \theta$</td>
<td>1</td>
<td>$-\sqrt{3}/2$</td>
<td>$-1/2$</td>
<td>$\sqrt{2}/2$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>0</td>
<td>$1/2$</td>
<td>$\sqrt{3}/2$</td>
<td>$\sqrt{2}/2$</td>
<td>$-\sqrt{3}/2$</td>
</tr>
<tr>
<td>$\tan \theta$</td>
<td>UND</td>
<td>$-\sqrt{3}$</td>
<td>$-1/\sqrt{3}$</td>
<td>1</td>
<td>$-1/\sqrt{3}$</td>
</tr>
<tr>
<td>$\cot \theta$</td>
<td>0</td>
<td>$-1/\sqrt{3}$</td>
<td>$-\sqrt{3}$</td>
<td>1</td>
<td>$-\sqrt{3}$</td>
</tr>
<tr>
<td>$\sec \theta$</td>
<td>UND</td>
<td>2</td>
<td>$2/\sqrt{3}$</td>
<td>$\sqrt{2}$</td>
<td>$-2/\sqrt{3}$</td>
</tr>
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<td>$\csc \theta$</td>
<td>1</td>
<td>$-2/\sqrt{3}$</td>
<td>$-2$</td>
<td>$\sqrt{2}$</td>
<td>2</td>
</tr>
</tbody>
</table>
Exercise 1.3.31. Use the addition formulas to derive the identity 
\[ \cos \left( x - \frac{\pi}{2} \right) = \sin x. \]

Solution. We have the formula \( \cos(A - B) = \cos A \cos B + \sin A \sin B \), so with \( A = x \) and \( B = \pi/2 \) we have
\[
\cos(x - \pi/2) = \cos x \cos \pi/2 + \sin x \sin \pi/2 = \cos x(0) + \sin x(1) = \sin x.
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\( \square \)

Notice that \( x \) and \( x - \pi/2 \) are complementary angles since 
\( (x) + (x - \pi/2) = \pi/2 \). So this exercise shows that the sine of an angle equals the cosine of its complement; *this* is why cosine is called “cosine.”
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Example 1.3.A. For any angle $\theta$ measured in radians, we have $-|\theta| \leq \sin \theta \leq |\theta|$ and $-|\theta| \leq 1 - \cos \theta \leq |\theta|$.

Solution. As in Figure 1.47, we put $\theta$ in standard position. Since the circle is a unit circle (that is, $r = 1$), then $|\theta|$ equals the length of the circular arc $AP$. 
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Solution (continued). We see from the figure that the length of line segment $AP$ is less than or equal to $|\theta|$. Triangle $APQ$ is a right triangle with sides of length $QP = |\sin \theta|$ and $AQ = 1 - \cos \theta$. So by the Pythagorean Theorem (and the fact that $AP \leq |\theta|$) we have

$$\sin^2 \theta + (1 - \cos \theta)^2 = (AP)^2 \leq \theta^2.$$ 

So we have both $\sin^2 \theta \leq \theta^2$ and $(1 - \cos \theta)^2 \leq \theta^2$. 

\[ \hspace{10cm} \]
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So we have both $\sin^2 \theta \leq \theta^2$ and $(1 - \cos \theta)^2 \leq \theta^2$. Taking square roots (and observing that the square root function is an increasing function so that it preserves inequalities),

$$\sqrt{\sin^2 \theta} \leq \sqrt{\theta^2} \quad \text{and} \quad \sqrt{(1 - \cos \theta)^2} \leq \sqrt{\theta^2},$$

or $|\sin \theta| \leq |\theta|$ and $|1 - \cos \theta| \leq |\theta|$. These two inequalities imply that $-|\theta| \leq \sin \theta \leq |\theta|$ and $-|\theta| \leq 1 - \cos \theta \leq |\theta|$, as claimed (see Appendix A.1. Real Numbers and the Real Line where intervals are related to absolute values). □
Example 1.3.A (continued)

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Exercise 1.3.68

Exercise 1.3.68. The general sine curve is

$$f(x) = A \sin \left( \frac{2\pi}{B} (x - C) \right) + D.$$  

For $y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2}$ identify $A$, $B$, $C$, and $D$ and sketch the graph.

Solution. First we write

$$y = \frac{1}{2} \sin(\pi x - \pi) + \frac{1}{2} = \frac{1}{2} \sin (\pi(x - 1)) + \frac{1}{2} = \frac{1}{2} \sin \left( \frac{2\pi}{2} (x - 1) \right) + \frac{1}{2}.$$  

We have $A = 1/2$, $B = 2$, $C = 1$, and $D = 1/2$. Now $A$ is the amplitude, $B$ is the period, $C$ is the horizontal shift, and $y = D$ is the axis. ...
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Solution. First we write

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We have \( A = 1/2, B = 2, C = 1, \) and \( D = 1/2. \) Now \( A \) is the amplitude, \( B \) is the period, \( C \) is the horizontal shift, and \( y = D \) is the axis. . . .
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