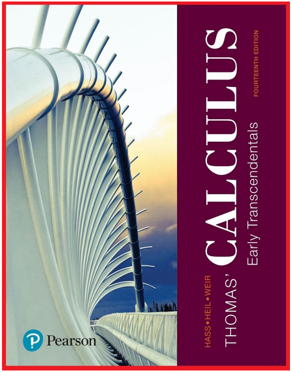
Calculus 1

Chapter 1. Functions

1.5. Exponential Functions—Examples and Proofs



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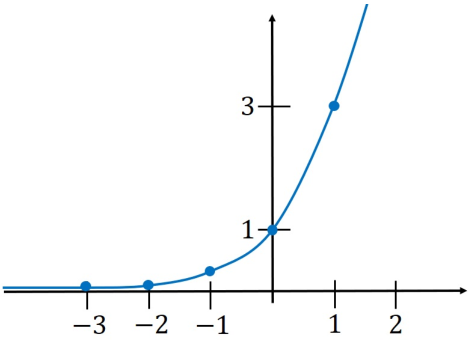

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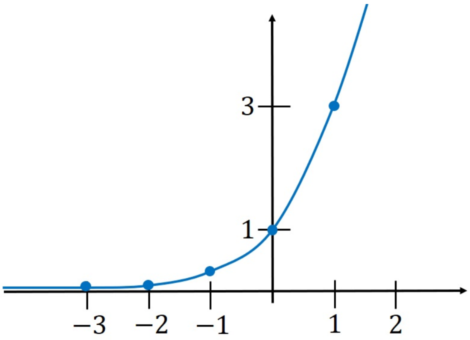

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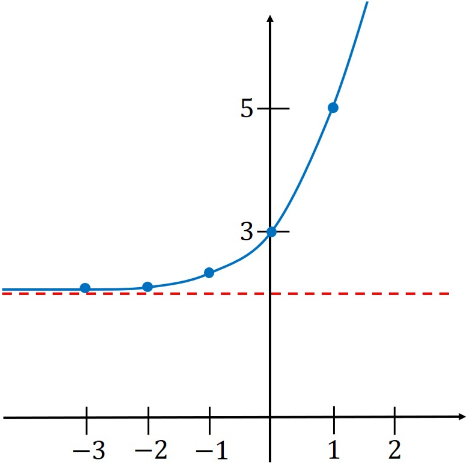

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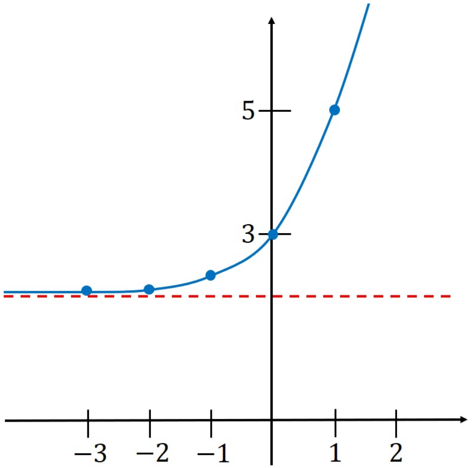

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The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year (that is, $k = 0.0275$). **(a)** Estimate the population in 1915 and 1940.

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Solution. With $r = 1.2 \times 10^{-4}$ and $t = 866$, we have $u = y_0 e^{(-0.00012)(866)} = y_0 e^{-0.10392} \approx (0.9013) y_0$. So the percent of carbon-14 present at this time is $(0.9013 y_0)/y_0 \times 100\% = \boxed{90.13\%}$.

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