Chapter 1. Functions
1.5. Exponential Functions—Examples and Proofs
<table>
<thead>
<tr>
<th></th>
<th>Exercise 1.5.2(a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Exercise 1.5.8(a)</td>
</tr>
<tr>
<td>3</td>
<td>Exercise 1.5.12</td>
</tr>
<tr>
<td>4</td>
<td>Exercise 1.5.30(a). Population Growth</td>
</tr>
<tr>
<td>5</td>
<td>Example 1.5.4</td>
</tr>
</tbody>
</table>
Exercise 1.5.2(a)

Exercise 1.5.2(a). Plot several points and sketch the graph of \( y = 3^x \).

Solution. Consider the function values:

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
x & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
f(x) & 3^{-3} = 1/27 & 3^{-2} = 1/9 & 3^{-1} = 1/3 & 3^0 = 1 & 3^1 = 3 & 3^2 = 9 & 3^3 = 27 \\
\hline
\end{array}
\]
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Exercise 1.5.8(a). Sketch the shifted exponential curve $y = 3^x + 2$.

Solution. We simply shift the graph of $y = 3^x$ from Exercise 1.5.2(a) up by 2 units (because of the "+2"). Notice that the graph of $y = 3^x + 2$ has a horizontal asymptote of $y = 2$. 
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![Graph of exponential function with horizontal asymptote at y=2]
Exercise 1.5.12

**Exercise 1.5.12.** Use the Law of Exponents (Theorem 1.5.A) to simplify $9^{1/3}9^{1/6}$.

**Solution.** We have by the Rules for Exponents (Theorem 1.5.A) that $a^x a^y = a^{x+y}$. 
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Solution. We have by the Rules for Exponents (Theorem 1.5.A) that $a^x a^y = a^{x+y}$.

So with $a = 9$, $x = 1/3$, and $y = 1/6$, we have $9^{1/3}9^{1/6} = 9^{1/3+1/6} = 9^{1/2} = \sqrt{9} = 3$. □
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Exercise 1.5.30(a). Population Growth.
The population of Silver Run in the year 1890 was 6250. Assume the population increased at a rate of 2.75% per year (that is, $k = 0.0275$). (a) Estimate the population in 1915 and 1940.

Solution. The population size is given by $y = y_0 e^{kt}$ where $k = 0.0275$, $y_0 = 6250$, and $t$ is time measured in years after 1890.
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In 1940, \( t = 1940 - 1890 = 50 \) and so the population is \( y = 6250e^{0.0275(50)} = 6250e^{1.375} \approx 24,719 \). □
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Example 1.5.4. Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder of the atom reforming to make an atom of some new element. For example, radioactive carbon-14 decays into nitrogen; radium eventually decays into lead. If \( y_0 \) is the number of radioactive nuclei present at time zero, the number still present at any later time \( t \) will be \( y = y_0 e^{-rt} \) where \( r > 0 \). The number \( r \) is the decay rate of the radioactive substance. For carbon-14, the decay rate has been determined experimentally to be about \( r = 1.2 \times 10^{-4} \) when \( t \) is measured in years. Predict the percent of carbon-14 present after 866 years have elapsed.

Solution. With \( r = 1.2 \times 10^{-4} \) and \( t = 866 \), we have

\[
y = y_0 e^{-0.00012(866)} = y_0 e^{-0.10392} \approx (0.9013)y_0.
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So the percent of carbon-14 present at this time is \( \frac{0.9013y_0}{y_0} \times 100\% = 90.13\% \).
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u = y_0 e^{(-0.00012)(866)} = y_0 e^{-0.10392} \approx (0.9013)y_0.\]
So the percent of carbon-14 present at this time is \( (0.9013y_0)/y_0 \times 100% = 90.13% \).