Example 2.6.1(a)

Example 2.6.1(a). Prove that \( \lim_{x \to \infty} \frac{1}{x} = 0 \).

**Proof.** First, notice that with \( P = \infty \) we have that the domain of \( f \) contains the interval \( (P, \infty) = (0, \infty) \). Let \( \varepsilon > 0 \) be given. [We must find a number \( M \) such that for all \( x > M \) implies \( \left| \frac{1}{x} - 0 \right| = \left| \frac{1}{x} \right| < \varepsilon \). The implication will hold if \( M = 1/\varepsilon \) or any larger positive number (see Figure 2.50).]

Suppose \( x > M = 1/\varepsilon \) (notice then that \( x \) is positive). This implies \( 0 < 1/x < 1/M = \varepsilon \), or \( 0 < \frac{1}{x} = \frac{1}{x} - 0 = |f(x) - L| < \varepsilon \).

Therefore \( \lim_{x \to \infty} \frac{1}{x} = 0 \), as claimed.

![Figure 2.50](image)

Exercise 2.6.14

Exercise 2.6.14. For the rational function \( f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7} \), find the limit as (a) \( x \to \infty \), and (b) \( x \to -\infty \). Justify your computations with Theorem 2.12.

**Solution.** We can evaluate both limits by the same process. We have

\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{2x^3 + 7}{x^3 - x^2 + x + 7}
\]

by the definition of \( f \)

\[
= \lim_{x \to \pm \infty} \frac{2x^3 + 7}{x^3 - x^2 + x + 7}(x^3)^{-1}
\]

dividing the numerator and denominator by the highest power of \( x \) in the denominator

\[
= \lim_{x \to \pm \infty} \frac{(2x^3 + 7)/x^3}{(x^3 - x^2 + x + 7)/x^3}
\]

\[
= \lim_{x \to \pm \infty} \frac{2x^3/x^3 + 7/x^3}{x^3/x^3 - x^2/x^3 + x/x^3 + 7/x^3}
\]

Solution (continued).

\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{2x^3/x^3 + 7/x^3}{x^3/x^3 - x^2/x^3 + x/x^3 + 7/x^3}
\]

\[
= \lim_{x \to \pm \infty} \frac{2 + 7/x^3}{-1 + 1/x + 1/x^2 + 7/x^3}
\]

since \( x \to \pm \infty \) then we can assume that \( x \neq 0 \)

\[
= \lim_{x \to \pm \infty} \frac{2 + 7/x^3}{-1 + 1/x + 1/x^2 + 7/x^3}
\]

by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0

\[
= \lim_{x \to \pm \infty} \frac{2 + lim_{x \to \pm \infty} 7/x^3}{lim_{x \to \pm \infty} 1 - lim_{x \to \pm \infty} 1/x + lim_{x \to \pm \infty} 1/x^2 + lim_{x \to \pm \infty} 7/x^3}
\]

by the Sum and Difference Rules (Theorem 2.12(1 and 2))

\[
= \lim_{x \to \pm \infty} \frac{2 + 7 lim_{x \to \pm \infty} 1/x^3}{lim_{x \to \pm \infty} 1 - lim_{x \to \pm \infty} 1/x + lim_{x \to \pm \infty} 1/x^2 + 7 lim_{x \to \pm \infty} 1/x^3}
\]

Exercise 2.6.14 (continued 1)
Exercise 2.6.36

Evaluate \( \lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} \) by dividing the numerator and denominator by the (effective) highest power of \( x \) in the denominator. Justify your computations with Theorem 2.12.

Solution. We have a square root of \( x^6 \) in the denominator, so the “effective” highest power of \( x \) in the denominator is 3 (think what happens when \( x \) is really large: \( x^6 + 9 \) is about the same size as \( x^6 \) and \( \sqrt{x^6} + 9 \) is about the same size as \( x^3 \)). We have

\[
\lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} = \lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} \left( \frac{1/x^3}{1/x^3} \right) \text{ by dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}
\]

\[
= \lim_{x \to \infty} \frac{(4 - 3x^3)/x^3}{(\sqrt{x^6} + 9)/x^6}
\]

Exercise 2.6.36 (continued 1)

Solution (continued).

\[
= \lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} \frac{1/x^3}{x^3} = \lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} \frac{1}{x^6}
\]

since \( \sqrt{x^6} = |x|^3 = -x^3 \) for \( x \) negative

\[
= \lim_{x \to \infty} \frac{4 - 3x^3}{\sqrt{x^6} + 9} \frac{1}{x^6} = \lim_{x \to \infty} \frac{4/x^3 - 3/x^3}{\sqrt{x^6} + 9/x^6}
\]

\[
= \lim_{x \to \infty} \frac{4/x^3 - 3}{\sqrt{1 + 9/x^6}} \text{ since } x \to -\infty \text{ then we can assume that } x \neq 0
\]

\[
= \lim_{x \to -\infty} \frac{4/x^3 - 3}{\sqrt{1 + 9/x^6}} \text{ by the Quotient Rule (Theorem 2.12(5)), assuming the denominator is not 0}
\]

Exercise 2.6.36 (continued 2)

Solution (continued).

\[
= \lim_{x \to -\infty} \frac{4/x^3 - 3}{\sqrt{1 + 9/x^6}} - \lim_{x \to -\infty} \frac{1}{\sqrt{1 + 9/x^6}} - \lim_{x \to -\infty} \frac{1}{\sqrt{1 + 9/x^6}} \text{ by the Difference Rule and the Constant Multiple Rule, Theorem 2.12(2 and 4)}
\]

\[
= \lim_{x \to -\infty} \frac{4/x^3 - 3}{\sqrt{1 + 9/x^6}} \text{ by the Root Rule, Theorem 2.12(7)}
\]

\[
= \lim_{x \to -\infty} (4/x^3 - 3) \text{ by the Sum Rule and Constant Multiple Rule, Theorem 2.12(1 and 4)}
\]

\[
= 4 \lim_{x \to -\infty} (1/x^3) - \lim_{x \to -\infty} (1/x^6) \text{ by the Power Rule, Theorem 2.12(6)}
\]
Exercise 2.6.36. Evaluate \( \lim_{x \to \pm\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} \) by dividing the numerator and denominator by the (effective) highest power of \( x \) in the denominator.

Solution (continued).
\[
\lim_{x \to \pm\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}} = \frac{4 \lim_{x \to \pm\infty} \frac{1}{x^3} - 3}{\sqrt{\lim_{x \to \pm\infty} (1) + 9 \lim_{x \to \pm\infty} \frac{1}{x^6}}} = \frac{4(0)^3 - (3)}{-\sqrt{(1) + 9(0)^6}} = \frac{-3}{1} = -3.
\]

Exercise 2.6.68

Exercise 2.6.68. Find the horizontal asymptote(s) of the graph of \( y = \frac{2x}{x + 1} \). Justify your computations with Theorem 2.12.

Solution. By definition of horizontal asymptote, we are led to consider \( \lim_{x \to \pm\infty} \frac{2x}{x + 1} \). We have
\[
\lim_{x \to \pm\infty} \frac{2x}{x + 1} = \lim_{x \to \pm\infty} 2 \left( \frac{1}{x} \right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}
\]
\[
= \lim_{x \to \pm\infty} \frac{(2x)(1/x)}{(x + 1)(1/x)} = \lim_{x \to \pm\infty} \frac{2x}{x + 1/x}.
\]
Since \( x \to \pm\infty \) then we can assume that \( x \neq 0 \).

Exercise 2.6.68 (continued)

Solution (continued).
\[
\lim_{x \to \pm\infty} \frac{2x}{x + 1} = \lim_{x \to \pm\infty} \frac{2}{1 + 1/x}
= \lim_{x \to \pm\infty} \frac{2}{\frac{1}{x} + 1/x} \text{ by the Quotient Rule}
\]
(\text{Theorem 2.12(5)}, \text{ assuming the denominator is not 0})
\[
= \frac{2}{\lim_{x \to \pm\infty} (1) + \lim_{x \to \pm\infty} (1/x)} \text{ by the Sum Rule,}
\]
\text{Theorem 2.12(1)}
\[
= \frac{2}{(1) + (0)} = 2 \text{ by Example 2.6.1.}
\]
Since \( \lim_{x \to \pm\infty} \frac{2x}{x + 1} = 2 \), then \( y = 2 \) is a horizontal asymptote of the graph of \( y = \frac{2x}{x + 1} \).

Example 2.6.4

Example 2.6.4. Find the horizontal asymptote(s) of the graph of \( y = \frac{x^3 - 2}{|x|^3 + 1} \). Justify your computations with Theorem 2.12.

Solution. A rational function can have only one horizontal asymptote. Since we are not given a rational function (because of the presence of the absolute value), then we consider \( x \to \infty \) and \( x \to -\infty \) separately. We divide the numerator and denominator by the highest (effective) power of \( x \) in the denominator. We have
\[
\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} \left( \frac{1/x^3}{1/x^3} \right)
= \lim_{x \to \infty} \frac{(x^3 - 2)(1/x^3)}{(|x|^3 + 1)(1/x^3)} = \lim_{x \to \infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3} \text{ since } x \to \infty \text{ then we can assume that } x \text{ is positive so that } |x|^3 = x^3.
\]
Example 2.6.4 (continued 1)

Solution (continued).
\[
\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to \infty} \frac{x^3/x^3 - 2/x^3}{x^3/x^3 + 1/x^3}
\]
\[
= \lim_{x \to \infty} \frac{1 - 2/x^3}{1 + 1/x^3} \quad \text{since } x \to \infty \text{ then we can assume that } x \neq 0
\]
\[
= \lim_{x \to \infty} (1 - 2/x^3) \cdot \lim_{x \to \infty} (1 + 1/x^3)
\]
\[
= \lim_{x \to \infty} (1 - 2/x^3) \quad \text{by the Quotient Rule}
\]
\[
= \lim_{x \to \infty} (1 + 1/x^3)
\]
\[
(\text{Theorem 2.12(5)), assuming the denominator is not 0}
\]
\[
= \lim_{x \to \infty} (1) - \lim_{x \to \infty} (2/x^3) \quad \text{by the Sum Rule}
\]
\[
= \lim_{x \to \infty} (1) + \lim_{x \to \infty} (1/x^3)
\]
\[
\text{and the Difference Rule, Theorem 2.12(1 and 2)}
\]

Example 2.6.4 (continued 2)

Solution (continued).
\[
\lim_{x \to \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to \infty} (1) - 2 (\lim_{x \to \infty} (1/x))^3 \quad \text{by the Constant Mult. Rule and the Power Rule, Theorem 2.12(4 and 6)}
\]
\[
= \frac{(1) - 2(0)^3}{(1) + (0)^3} = 1 \quad \text{by Example 2.6.1(a)}.
\]

So the graph of \( y = \frac{x^3 - 2}{|x|^3 + 1} \) has a \[ \text{horizontal asymptote of } y = 1 \text{ as } x \to \infty. \]

Example 2.6.4 (continued 3)

Solution (continued). The computation is similar for \( x \to -\infty \), except that for \( x \) negative we have \( |x|^3 = -x^3 \). We have
\[
\lim_{x \to -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to -\infty} \frac{x^3 - 2}{|x|^3 + 1 (1/x^3)}
\]
\[
= \lim_{x \to -\infty} \frac{(x^3 - 2)(1/x^3)}{|x|^3 + 1 (1/x^3)} = \lim_{x \to -\infty} \frac{x^3/x^3 - 2/x^3}{|x|^3 + 1/x^3} \quad \text{since}
\]
\[
\lim_{x \to -\infty} x \to -\infty \text{ then we can assume that } x \text{ is negative}
\]
\[
\text{so that } |x|^3 = -x^3
\]
\[
= \lim_{x \to -\infty} \frac{1 - 2/x^3}{1 + 1/x^3} \quad \text{since } x \to -\infty \text{ then we can assume that } x \neq 0
\]
\[
= \lim_{x \to -\infty} (1 - 2/x^3) \quad \text{by the Quotient Rule}
\]
\[
= \lim_{x \to -\infty} (1 + 1/x^3)
\]
\[
(\text{Theorem 2.12(5)), assuming the denominator is not 0}
\]

Example 2.6.4 (continued 4)

Solution (continued).
\[
\lim_{x \to -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \to -\infty} (1) - \lim_{x \to -\infty} (2/x^3) \quad \text{by the Sum Rule}
\]
\[
= \lim_{x \to -\infty} (1) - 2 (\lim_{x \to -\infty} (1/x))^3 \quad \text{by the Sum Rule}
\]
\[
= \lim_{x \to -\infty} (1) + \lim_{x \to -\infty} (1/x^3)
\]
\[
\text{and the Difference Rule, Theorem 2.12(1 and 2)}
\]
\[
= \frac{(1) - 2(0)^3}{(-1) + (0)^3} = -1 \quad \text{by Example 2.6.1(b)}.
\]

So the graph of \( y = \frac{x^3 - 2}{|x|^3 + 1} \) has a \[ \text{horizontal asymptote of } y = -1 \text{ as } x \to -\infty. \]
Example 2.6.5

Example 2.6.5. Use the formal definition to prove \( \lim_{x \to -\infty} e^x = 0 \). Notice that this implies that \( y = 0 \) is a horizontal asymptote of \( y = e^x \).

Proof. First, the domain of \( f(x) = e^x \) is all of the real numbers \( \mathbb{R} \), so it is defined on an interval of the form \((-\infty, P)\) (for any \( P \)). Next, let \( \varepsilon > 0 \). Choose \( N = \ln \varepsilon \). If \( x < N = \ln \varepsilon \) then \( e^x < e^{\ln \varepsilon} = \varepsilon \) since \( e^x \) is an increasing function on the real numbers.
That is, if \( x < N \) then
\[
|f(x) - 0| = |e^x - 0| = e^x < \varepsilon.
\]
Therefore, by definition,
\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} e^x = 0,
\]
as claimed. \( \Box \)

Note. The choice of \( N = \ln \varepsilon \) makes sense if we consider the graph of \( y = e^x \):

Example 2.6.8

Example 2.6.8. Use the Sandwich Theorem to find the horizontal asymptote of the curve \( y = 2 + \frac{\sin x}{x} \).

Solution. First, \(-1 \leq \sin x \leq 1\) for all real numbers. Let \( g(x) = 2 - 1/x \), \( f(x) = 2 + \frac{\sin x}{x} \), and \( h(x) = 2 + 1/x \). Then \( g(x) \leq f(x) \leq h(x) \) for all real numbers, except 0, and so these inequalities hold on \((-\infty, P) = (-\infty, 0)\) and \((P, \infty) = (0, \infty)\). Now
\[
\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (2 - 1/x) = 2 - 0 = 2 = L
\]
and
\[
\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} (2 + 1/x) = 2 + 0 = 2 = L, \text{ by Example 2.6.1.}
\]
So by Theorem 2.6.2,
\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left( 2 + \frac{\sin x}{x} \right) = 2. \text{ Therefore,}
\]
\[
y = 2 \text{ is a horizontal asymptote of the graph of } y = 2 + \frac{\sin x}{x}. \Box
\]

Exercise 2.6.92

Exercise 2.6.92. Evaluate (carefully!) \( \lim_{x \to \infty} \left( \sqrt{x^2 + x} - \frac{\sqrt{x^2 - x}}{x} \right) \).
Justify your computations.

Solution. We multiply by the conjugate of the given expression divided by itself (which is defined for \( x \) “sufficiently large,” namely \( x \geq 1 \)) in order to produce a quotient and try to use some of the techniques already introduced. We have
\[
\lim_{x \to \infty} \left( \sqrt{x^2 + x} - \frac{\sqrt{x^2 - x}}{x} \right) = \lim_{x \to \infty} \frac{\sqrt{x^2 + x} - \frac{\sqrt{x^2 - x}}{x}}{\sqrt{x^2 + x} - \frac{\sqrt{x^2 - x}}{x}}
\]
\[
= \lim_{x \to \infty} \frac{(\sqrt{x^2 + x})^2 - (\sqrt{x^2 - x})^2}{\sqrt{x^2 + x} + \frac{\sqrt{x^2 - x}}{x}} = \lim_{x \to \infty} \frac{x^2 + x - (x^2 - x)}{\sqrt{x^2 + x} + \frac{\sqrt{x^2 - x}}{x}}
\]
\[
= \lim_{x \to \infty} \frac{x^2 + x}{\sqrt{x^2 + x} + \frac{\sqrt{x^2 - x}}{x}}.
\]
Exercise 2.6.92 (continued 1)

Solution (continued).

\[
\lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{2x}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} \left( \frac{1/x}{1/x} \right) \text{ dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}
\]

\[
= \lim_{x \to \infty} \frac{(2x)/x}{\sqrt{x^2 + x} + \sqrt{x^2 - x} / x} = \lim_{x \to \infty} \frac{(2x)/x}{\sqrt{x^2 + x} + \sqrt{x^2 - x} / \sqrt{x^2}}
\]

since \( x \to \infty \) then we can assume that \( x \neq 0 \)

so that \( \sqrt{x^2} = |x| = x \)

\[
= \lim_{x \to \infty} \frac{(2x)/x}{\sqrt{x^2 + x} / \sqrt{x^2} + \sqrt{x^2 - x} / \sqrt{x^2} / \sqrt{x^2}}
\]

Exercise 2.6.92 (continued 2)

Solution (continued).

\[
\lim_{x \to \infty} \frac{(2x)/x}{\sqrt{(x^2 + x)/x^2} + \sqrt{(x^2 - x)/x^2}} = \lim_{x \to \infty} \frac{(2x)/x}{\sqrt{x^2/x^2 + x/x^2} + \sqrt{x^2/x^2 - x/x^2}}
\]

\[
= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \sqrt{1 - 1/x}} / \sqrt{1 - 1/x}} \text{ since } x \to \infty \text{ then we can assume that } x \neq 0
\]

\[
= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \sqrt{1 - 1/x}}} \text{ by the Quotient Rule}
\]

Theorem 2.12(5), assuming the denominator is not 0

\[
= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \lim_{x \to \infty} \sqrt{1 - 1/x}}} \text{ by the Sum Rule, Theorem 2.12(1)}
\]

Exercise 2.6.92 (continued 3)

Solution (continued).

\[
= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \sqrt{1 - 1/x}}} = \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \sqrt{1 - 1/x}}}
\]

Theorem 2.12(7) (notice that both \( 1 + 1/x \) and \( 1 - 1/x \) are nonnegative for \( x \geq 1 \))

\[
= \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x + \sqrt{1 - 1/x} + \sqrt{1 - 1/x}}}
\]

by the Sum and Difference Rules, Theorem 2.12(1 and 2)

\[
= \frac{2}{\sqrt{1 + 0} + \sqrt{1 - 0}} = \frac{2}{1 + 1} = 1.
\]

Exercise 2.6.108

Exercise 2.6.108. Consider the rational function \( y = \frac{x^2 - 1}{2x + 4} \). Find the oblique asymptote.

Solution. First, we perform long division to get:

\[
\begin{array}{c|ccccc}
2x+4 & x^2 & -1 \\
\hline
 & x^2 & +2x \\
 & -2x & -1 \\
 & -2x & -4 \\
\end{array}
\]

So \( y = \frac{x^2 - 1}{2x + 4} = \frac{x}{2 - 1 + \frac{3}{2x + 4}} \) where \( x/2 - 1 \) is a linear term. If we show that \( \lim_{x \to \pm \infty} \frac{3}{2x + 4} = 0 \) then we can conclude that \( y = x/2 - 1 \) is the oblique asymptote for the graph of \( y = (x^2 - 1)/(2x + 4) \).
Exercise 2.6.108 (continued 1)

Solution (continued). Next,
\[
\lim_{x \to \pm \infty} \frac{3}{2x + 4} = \lim_{x \to \pm \infty} \frac{3}{2x + 4} \left( \frac{1}{x} \right) \text{ dividing the numerator and denominator by the highest power of } x \text{ in the denominator}
\]
\[
= \lim_{x \to \pm \infty} \frac{3}{(2x + 4)(1/x)} = \lim_{x \to \pm \infty} \frac{3/x}{2x/x + 4/x}
\]
\[
= \lim_{x \to \pm \infty} \frac{3/x}{2 + 4/x} \text{ since } x \to \pm \infty \text{ then we can assume that } x \neq 0
\]
\[
= \frac{3}{2 \cdot 4} \text{ by the Sum, Constant, Multiple and Quotient Rules, Theorem 2.12(1, 4, & 5)}
\]
\[
= \frac{3(0)}{(2) + 4(0)} = 0 \text{ by Example 2.6.1.}
\]

Exercise 2.6.108 (continued 2)

Exercise 2.6.108. Consider the rational function \( y = \frac{x^2 - 1}{2x + 4} \). Find the oblique asymptote.

Solution (continued). Since \( y = \frac{x^2 - 1}{2x + 4} = \frac{x}{2} - 1 + \frac{3}{2x + 4} \) and
\[
\lim_{x \to \pm \infty} \frac{3}{2x + 4} = 0, \text{ then } y = \frac{x}{2} - 1 \text{ is an oblique asymptote of the graph of } y = \frac{x^2 - 1}{2x + 4}. \text{ Notice that the function } f(x) = \frac{x^2 - 1}{2x + 4} \text{ is not defined at } x = -2. \text{ With } y = \frac{x^2 - 1}{2x + 4} = \frac{x}{2} - 1 + \frac{3}{2x + 4}, \text{ the term } \frac{3}{2x + 4} \text{ is positive for } x \text{ large and positive, and is negative for } x \text{ large and negative. So the graph of } y = \frac{x^2 - 1}{2x + 4} \text{ lies above the oblique asymptote } y = \frac{x}{2} - 1 \text{ for } x \text{ large and positive, and lies below the oblique asymptote for } x \text{ large and negative.}

Exercise 2.6.108 (continued 3)

Solution (continued). A crude graph of \( y = \frac{x^2 - 1}{2x + 4} \) which reflects the oblique asymptote (but does not reflect other subtle details of the graph) is as follows (we’ll explore the graph in more detail later):

Example 2.6.B

Example 2.6.B. For \( n \) a positive even integer, prove that \( \lim_{x \to 0} \frac{1}{x^n} = \infty. \)

Solution. First, \( f(x) = 1/x^n \) is defined for all \( x \) except 0, so there is an open interval containing \( c = 0 \) on which \( f \) is defined, except at \( c = 0 \) itself (say the interval \((-1, 1))\). Let \( B \) be a positive real number.

Choose \( \delta = 1/B^{1/n} \). Then for
\[ 0 < |x - c| = |x| < \delta = 1/B^{1/n} \text{ we have } \frac{1}{|x|} > B^{1/n} \text{ (since the function } 1/x \text{ is decreasing for } x > 0 \text{ and so } 1/|x|^n > B \text{ (since the function } x^n \text{ is increasing for } x \geq 0 \text{). Since } n \text{ is even, then } |x|^n = x^n \text{ and so we have } f(x) = 1/x^n = 1/|x|^n > B. \text{ So, by definition, } \lim_{x \to 0} \frac{1}{x^n} = \infty, \text{ as claimed.} \]
Exercise 2.6.54

Exercise 2.6.54. Consider \( f(x) = \frac{x}{x^2 - 1} \). Find (a) \( \lim_{x \to 1^+} f(x) \), (b) \( \lim_{x \to -1} f(x) \), (c) \( \lim_{x \to 1^+} f(x) \), and (d) \( \lim_{x \to -1} f(x) \).

Solution. First, \( f(x) = \frac{x}{x^2 - 1} \) is a rational function of the form \( f(x) = \frac{p(x)}{q(x)} \) where \( p(x) = x \) and \( q(x) = x^2 - 1 \).

(a) We have \( \lim_{x \to 1^+} p(x) = \lim_{x \to 1^+} x = 1 \neq 0 \) and \( \lim_{x \to 1^+} q(x) = \lim_{x \to 1^+} (x^2 - 1) = (1)^2 - 1 = 0 \), by Theorem 2.2 for one-sided limits. So by Dr. Bob’s Infinite Limits Theorem,
\[
\lim_{x \to 1^+} \frac{p(x)}{q(x)} = \lim_{x \to 1^+} \frac{x}{x^2 - 1} = \pm \infty; \text{ we just need to determine if the limit is } +\infty \text{ or } -\infty.\]
We do so by analyzing the sign of \( \frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \) for “appropriate” \( x \) (since \( x \to 1^+ \), then appropriate \( x \) are close to 1 and slightly greater than 1). For such \( x \), we have \( x \) is positive (in fact, \( x \) is “close to” 1), \( x - 1 \) is positive (since \( x \) is greater than 1; so \( x - 1 \) is positive and “close to” 0), and \( x + 1 \) is positive (in fact, \( x \) is “close to” 2).

Solution (continued). Combining the factors we can conclude the following little sign diagram (not an actual equation):
\[
\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \Rightarrow \begin{cases} (+) & \text{if } \frac{x}{x^2 - 1} > 0 \\ (-)(+) & \text{if } \frac{x}{x^2 - 1} < 0 \end{cases}
\]
Since we know
\[
\lim_{x \to 1^+} \frac{x}{x^2 - 1} = +\infty \text{ and we know for } x \text{ close to 1 and slightly less than 1 that } \frac{x}{x^2 - 1} \text{ is negative, then we conclude that } \lim_{x \to 1^+} \frac{x}{x^2 - 1} = -\infty. \square
\]

(b) We have \( \lim_{x \to 1^-} p(x) = \lim_{x \to 1^-} x = 1 \neq 0 \) and \( \lim_{x \to 1^-} q(x) = \lim_{x \to 1^-} (x^2 - 1) = (1)^2 - 1 = 0 \), by Theorem 2.2 for one-sided limits. So by Dr. Bob’s Infinite Limits Theorem,
\[
\lim_{x \to 1^-} \frac{p(x)}{q(x)} = \lim_{x \to 1^-} \frac{x}{x^2 - 1} = \pm \infty; \text{ we just need to determine if the limit is } +\infty \text{ or } -\infty.\]
We do so, again, by analyzing the sign of \( \frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \) for “appropriate” \( x \) (since \( x \to 1^- \), then appropriate \( x \) are close to 1 and slightly less than 1).

Exercise 2.6.54 (continued 2)

Exercise 2.6.54 (continued 3)

Solution (continued). For such \( x \), we have \( x \) is positive (in fact, \( x \) is “close to” 1), \( x - 1 \) is negative (since \( x \) is less than 1; so \( x - 1 \) is negative and “close to” 0), and \( x + 1 \) is positive (in fact, \( x \) is “close to” 2).

Combining the factors we can again conclude the following little sign diagram:
\[
\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \Rightarrow \begin{cases} (+) & \text{if } \frac{x}{x^2 - 1} > 0 \\ (-)(+) & \text{if } \frac{x}{x^2 - 1} < 0 \end{cases}
\]
Since we know
\[
\lim_{x \to 1^-} \frac{x}{x^2 - 1} = -\infty \text{ and we know for } x \text{ close to 1 and slightly less than 1 that } \frac{x}{x^2 - 1} \text{ is negative, then we conclude that } \lim_{x \to 1^-} \frac{x}{x^2 - 1} = -\infty. \square
\]

(c) We have \( \lim_{x \to 1^+} p(x) = \lim_{x \to 1^+} x = -1 \neq 0 \) and \( \lim_{x \to 1^+} q(x) = \lim_{x \to 1^+} (x^2 - 1) = (-1)^2 - 1 = 0 \), by Theorem 2.2 for one-sided limits. So by Dr. Bob’s Infinite Limits Theorem,
\[
\lim_{x \to 1^+} \frac{p(x)}{q(x)} = \lim_{x \to 1^+} \frac{x}{x^2 - 1} = \pm \infty; \text{ we just need to determine if the limit is } +\infty \text{ or } -\infty.
\]

Solution (continued). We analyze the sign of \( \frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \) for “appropriate” \( x \) (since \( x \to -1^+ \), then appropriate \( x \) are close to \(-1\) and slightly greater than \(-1\)). For such \( x \), we have \( x \) is negative (in fact, \( x \) is “close to” \(-1\), \( x - 1 \) is negative (in fact, \( x - 1 \) is “close to” \(-2\)), and \( x + 1 \) is positive (since \( x \) is greater than \(-1\); so \( x + 1 \) is positive and “close to” 0). Combining the factors we get the sign diagram:
\[
\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \Rightarrow \begin{cases} (-) & \text{if } \frac{x}{x^2 - 1} < 0 \\ (+)(+) & \text{if } \frac{x}{x^2 - 1} > 0 \end{cases}
\]
Since we know
\[
\lim_{x \to -1^-} \frac{x}{x^2 - 1} = +\infty \text{ and we know for } x \text{ close to } -1 \text{ and slightly greater than } -1 \text{ that } \frac{x}{x^2 - 1} \text{ is positive, then we conclude that } \lim_{x \to -1^-} \frac{x}{x^2 - 1} = +\infty. \square
\]
Exercise 2.6.54 (continued 4)

Solution (continued). (d) We have
\[
\lim_{x \to -1^-} p(x) = \lim_{x \to -1^-} x = -1 \neq 0 \quad \text{and} \\
\lim_{x \to -1^-} q(x) = \lim_{x \to -1^-} (x^2 - 1) = (-1)^2 - 1 = 0, \quad \text{by Theorem 2.2 for} \\
\text{one-sided limits. So by Dr. Bob's Infinite Limits Theorem,} \\
\lim_{x \to -1^-} \frac{p(x)}{q(x)} = \lim_{x \to -1^-} \frac{x}{x^2 - 1} = \pm \infty; \quad \text{we just need to determine if the} \\
\text{limit is } +\infty \text{ or } -\infty. \quad \text{We analyze the sign of} \\
\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \quad \text{for} \\
\text{"appropriate" } x \quad (\text{since } x \to -1^-, \text{ then appropriate } x \text{ are close to } -1 \text{ and} \\
\text{slightly less than } -1). \quad \text{For such } x, \text{ we have } x \text{ is negative (in fact, } x \text{ is} \\
\text{"close to" } -1), \quad x - 1 \text{ is negative (in fact, } x - 1 \text{ is "close to" } -2), \text{ and} \\
x + 1 \text{ is negative (since } x \text{ is less than } -1; \text{ so } x + 1 \text{ is positive and "close to" } 0). \quad \text{Combining the factors we get the sign diagram:} \\
\frac{x}{x^2 - 1} = \frac{x}{(x - 1)(x + 1)} \Rightarrow \frac{(-)}{(-)(-)} = -. \]

Exercise 2.6.70

Exercise 2.6.70. Consider \( y = f(x) = \frac{2x}{x^2 - 1} \). Find the domain, horizontal asymptote(s), vertical asymptotes, graph \( y = f(x) \) in such a way as to reflect the asymptotic behavior, and find the range of \( f \).

Solution. First, the domain of \( y = f(x) = \frac{2x}{x^2 - 1} = \frac{2x}{(x + 1)(x - 1)} \) is all real \( x \) except for \(-1 \) and \( 1 \); the domain is \((-\infty, -1) \cup (-1, 1) \cup (1, \infty)\). For the horizontal asymptote(s), we consider \( \lim_{x \to \pm \infty} f(x) \). We have
\[
\lim_{x \to \pm \infty} f(x) = \lim_{x \to \pm \infty} \frac{2x}{x^2 - 1} \\
= \lim_{x \to \pm \infty} \frac{2x}{x^2 - 1} \left( \frac{1/x^2}{1/x^2} \right) \quad \text{dividing the numerator and} \\
\text{denominator by the highest power} \\
of \text{ } x \text{ in the denominator}
\]

So \( y = 0 \) is a horizontal asymptote of \( y = \frac{2x}{x^2 - 1} \).
Exercise 2.6.70 (continued 2)

Solution (continued). Now \( f(x) = \frac{2x}{x^2 - 1} = \frac{2x}{(x + 1)(x - 1)} \) is a rational function with \( \lim_{x \to -1^-} 2x = -2 \neq 0 \), \( \lim_{x \to -1^+} 2x = -2 \neq 0 \), and \( \lim_{x \to 1^-} 2x = -2 \neq 0 \) (each by Theorem 2.2), so by Dr. Bob’s Infinite Limits Theorem (applied to rational functions), \( f \) has vertical asymptotes at \( x = -1 \) and \( x = 1 \). We explore the vertical asymptotes by taking one-sided limits to determine if the limit is \( +\infty \) or \( -\infty \). We analyze the sign of \( \frac{2x}{(x - 1)(x + 1)} \) for “appropriate” \( x \) in each case. For \( x \to -1^+ \), the appropriate \( x \) are close to \(-1\) and slightly greater than \(-1\). For such \( x \), we have \( 2x \) is negative (in fact, \( 2x \) is “close to” \(-2\)), \( x - 1 \) is negative (in fact, \( x - 1 \) is “close to” \(-2\)), and \( x + 1 \) is positive (since \( x \) is greater than \(-1\); so \( x + 1 \) is positive and “close to” \( 0\)). Combining the factors we get the sign diagram:

\[
\frac{2x}{x^2 - 1} = \frac{2x}{(x - 1)(x + 1)} \Rightarrow \frac{(-)(+)}{(-)(+)} = + \text{. So } \lim_{x \to -1^+} f(x) = \infty.
\]

Exercise 2.6.70 (continued 3)

Solution (continued). For \( x \to -1^- \), the appropriate \( x \) are close to \(-1\) and slightly less than \(-1\). For such \( x \), we have \( 2x \) is negative (2x is “close to” \(-2\)), \( x - 1 \) is negative (since \( x \) is less than \(-1\); so \( x - 1 \) is negative and “close to” \(-2\)), and \( x + 1 \) is negative (since \( x \) is less than \(-1\); so \( x + 1 \) is negative and “close to” \( 0\)). Combining the factors we get the sign diagram:

\[
\frac{2x}{x^2 - 1} = \frac{2x}{(x - 1)(x + 1)} \Rightarrow \frac{(-)}{(-)(+)} = - \text{. So } \lim_{x \to -1^-} f(x) = -\infty.
\]

Exercise 2.6.70 (continued 4)

Solution (continued). For \( x \to 1^- \), the appropriate \( x \) are close to \( 1 \) and slightly less than \( 1 \). For such \( x \), we have \( 2x \) is positive (2x is “close to” \( 2 \)), \( x - 1 \) is negative (since \( x \) is less than \( 1 \); so \( x - 1 \) is negative and “close to” \( 0 \)), and \( x + 1 \) is positive (\( x + 1 \) is “close to” \( 2 \)). Combining the factors we get the sign diagram:

\[
\frac{2x}{x^2 - 1} = \frac{2x}{(x - 1)(x + 1)} \Rightarrow \frac{(+)(+)}{(-)(+)} = - \text{. So } \lim_{x \to 1^-} f(x) = -\infty.
\]

We have the graph (notice the range is all real numbers):
Exercise 2.6.108 (again). Consider the rational function \( y = \frac{x^2 - 1}{2x + 4} \). Find all asymptotes and graph in a way that reflects the asymptotic behavior.

**Solution.** We saw above that the graph of \( y = \frac{x^2 - 1}{2x + 4} = \frac{x}{2} - 1 + \frac{3}{2x + 4} \) has \( y = \frac{x}{2} - 1 \) as an oblique asymptote as \( x \to \pm \infty \). We now explore vertical asymptotes. By the Quotient Rule, Theorem 2.1(5),

\[
\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 1}{2x + 4} = \frac{c^2 - 1}{2c + 4} \quad \text{for} \quad c \neq -2.
\]

So by definition, \( f \) is continuous on its domain \((-\infty, -2) \cup (-2, \infty)\). By Dr. Bob’s Infinite Limits Theorem (applied to rational function \( f \)), since 

\[
\lim_{x \to -2} x^2 - 1 = (-2)^2 - 1 = 3 \neq 0 \quad \text{and} \quad \lim_{x \to -2} 2x + 4 = x(-2) + 4 = 4 = 0
\]

(by Theorem 2.2), we see that \( \lim_{x \to -2 \pm} f(x) = \pm \infty \) and so the graph has a \text{vertical asymptote of} \( x = -2 \). We explore one-sided limits to see if the limits are \( \infty \) or \( -\infty \).

**Solution (continued).** Combining the factors we get the sign diagram:

\[
x^2 - 1 \quad \Rightarrow \quad (+) \quad = \quad -. \quad \text{So} \quad \lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 - 1}{2x + 4} = -\infty.
\]

So the graph is:

![Graph of function](image)

Exercise 2.6.80. Find a function \( g \) that satisfies the conditions 

\[
\lim_{x \to \infty} g(x) = 0, \quad \lim_{x \to -3^-} g(x) = -\infty, \quad \text{and} \quad \lim_{x \to -3^+} g(x) = \infty. \]

Graph \( y = g(x) \) in a way that reflects the asymptotic behavior.

**Solution.** Since we want \( \lim_{x \to -3} g(x) = -\infty \), then the graph of \( y = g(x) \) will have \( y = 0 \) as a horizontal asymptote. Since we want \( \lim_{x \to -3} g(x) = -\infty \) and \( \lim_{x \to -3^+} g(x) = \infty \), then the graph of \( y = g(x) \) has a vertical asymptote of \( x = 3 \). We try to find a rational function, 

\[
g(x) = \frac{p(x)}{q(x)},
\]

satisfying these conditions. If we make polynomial \( p \) of degree less than that of polynomial \( q \), then this will give (as we will check) the horizontal asymptote \( y = 0 \). If we have \( x - 3 \) in the denominator then we should get a vertical asymptote of \( x = 3 \) (unless we also have a factor of \( x - 3 \) in the numerator, which we will avoid). So we try \( p(x) = 1 \) (a polynomial of degree 0), \( q(x) = x - 3 \) (a polynomial of degree 1), and 

\[
g(x) = 1/(x - 3)
\]

(we may have to adjust the sign of \( g \) to get the proper one side limits at 3).
Exercise 2.6.80 (continued 1)

Solution (continued). We have

\[
\lim_{x \to \pm \infty} g(x) = \lim_{x \to \pm \infty} \frac{1}{x - 3} = \lim_{x \to \pm \infty} \frac{1/x}{1/x - 3/x} \quad \text{dividing the numerator and denominator by the effective highest power of } x \text{ in the denominator}
\]

\[
= \lim_{x \to \pm \infty} \frac{1/x}{(x - 3)/x} = \lim_{x \to \pm \infty} \frac{1/x}{1 - 3/x} \quad \text{since } x \to \pm \infty
\]

then we can assume that \(x \neq 0\)

\[
= \lim_{x \to \pm \infty} \frac{1/x}{1 - 3/x} \quad \text{by the Difference, Constant Mult., and Quotient Rules, Theorem 2.12(2,4,5)}
\]

\[
= \frac{0}{1 - 3(0)} = \frac{0}{1} = 0 \quad \text{by Example 2.6.1.}
\]

So \(y = 0\) and a horizontal asymptote of the graph of \(y = g(x)\), as desired.

Exercise 2.6.80 (continued 2)

Solution (continued). Since \(\lim_{x \to 3^{-}} 1 = 1 \neq 0\) and \(\lim_{x \to 3^{+}} x - 3 = 0\) (both by Theorem 2.2, say), then by Dr. Bob’s Infinite Limits Theorem (applied to rational functions) \(\lim_{x \to 3^{\pm}} g(x) = \pm \infty\). We consider one-sided limits (as required by the question).

For \(\lim_{x \to 3^{+}} g(x)\), we analyze the sign of \(\frac{1}{x - 3}\) for “appropriate” \(x\) (since \(x \to 3^{+}\), then appropriate \(x\) are close to 3 and slightly greater than 3). For such \(x\), we have 1 is positive and \(x - 3\) is positive (since \(x\) is greater than 3; so \(x - 3\) is positive and “close to” 0). Combining the factors we get the sign diagram:

\[
\frac{1}{x - 3} \Rightarrow (+) \Rightarrow (+) = +.
\]

So \(\lim_{x \to 3^{+}} g(x) = \lim_{x \to 3^{+}} \frac{1}{x - 3} = \infty\), as desired.

For \(\lim_{x \to 3^{-}} g(x)\), we analyze the sign of \(\frac{1}{x - 3}\) for “appropriate” \(x\) (since \(x \to 3^{-}\), then appropriate \(x\) are close to 3 and slightly less than 3). For such \(x\), we have 1 is positive and \(x - 3\) is negative (since \(x\) is less than 3; so \(x - 3\) is negative and “close to” 0).

Exercise 2.6.80 (continued 3)

Exercise 2.6.80. Find a function \(g\) that satisfies the conditions

\[
\lim_{x \to \pm \infty} g(x) = 0, \quad \lim_{x \to 3^{-}} g(x) = -\infty, \quad \text{and } \lim_{x \to 3^{+}} g(x) = \infty.
\]

Graph \(y = g(x)\) in a way that reflects the asymptotic behavior.

Solution (continued). Combining the factors we get the sign diagram:

\[
\frac{1}{x - 3} \Rightarrow (+) \Rightarrow (-) = -.
\]

So

\[
\lim_{x \to 3^{-}} g(x) = \lim_{x \to 3^{-}} \frac{1}{x - 3} = -\infty,
\]

as desired.

We then have the graph:

Exercise 2.6.102

Exercise 2.6.102. Use the formal definition of an infinite one-sided limit to prove that \(\lim_{x \to 2^{-}} \frac{1}{x - 2} = -\infty\).

Proof. Let \(f\) be a function defined on an interval \((a, c)\), where \(a < c\). We say that \(f(x)\) approaches negative infinity as \(x\) approaches \(c\) from the left, and we write \(\lim_{x \to c^{-}} f(x) = -\infty\), if for every negative real number \(-B\) there exists a corresponding \(\delta > 0\) such that for all \(x\)

\[
c - \delta < x < c \implies f(x) < -B.
\]

(This is the solution to Exercise 2.6.99(c).) Notice that \(f(x) = 1/((x - 2)\) is defined on the interval \((-\infty, 2)\) (here, \(c = 2\)). Let \(-B\) be any negative real number. Choose \(\delta = 1/B > 0\). If \(2 - \delta < x < 2\) then \(-\delta < x < 2 < 0\) and \(-1/\delta > 1/(x - 2)\), since \(1/x\) is a decreasing function for negative input values. Now \(\delta = 1/B\) so \(1/\delta = B\) and \(-1/\delta = -B\). So \(2 - \delta < x < 2\) implies \(f(x) = 1/(x - 2) < -B\). Therefore, by the definition above, \(\lim_{x \to 2^{-}} 1/(x - 2) = -\infty\). □