Exercise 3.1.7

Exercise 3.1.7. Find an equation for the tangent line to the curve \( y = 2\sqrt{x} \) at the point \((1, 2)\). Then sketch the curve and tangent line together.

**Solution.** With \( y = f(x) = 2\sqrt{x} \) and \( P(x_0, f(x_0)) = (1, 2) \), we have the slope of the curve \( y = f(x) \) as

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2\sqrt{1}}{h} = \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2}{h} = \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2}{h} \left( \frac{2\sqrt{1 + h} + 2}{2\sqrt{1 + h} + 2} \right) \text{ multiplying by a form of 1 involving the conjugate}
\]

\[
= \lim_{h \to 0} \frac{(2\sqrt{1 + h} - 2)(2\sqrt{1 + h} + 2)}{h(2\sqrt{1 + h} + 2)} = \lim_{h \to 0} \frac{(2\sqrt{1 + h})^2 - (2)^2}{h(2\sqrt{1 + h} + 2)} = \lim_{h \to 0} \frac{4(1 + h) - 4}{h(2\sqrt{1 + h} + 2)} = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1 + h} + 2)} = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1 + h} + 2)}
\]

So the desired tangent line has slope \( m = 1 \) and passes through the point \((x_1, y_1) = (1, 2)\). By the point-slope formula, \( y - y_1 = m(x - x_1) \), the tangent line is \( y - 2 = (1)(x - 1) \) or \( y = x + 1 \).
Exercise 3.1.12.

Exercise 3.1.12. Find the slope of the graph of function \( f(x) = x - 2x^2 \) at the point \((1, -1)\). Then find an equation for the line tangent to the graph there.

Solution. With \( y = f(x) = x - 2x^2 \) and \( P(x_0, f(x_0)) = (1, -1) \), we have the slope of the curve \( y = f(x) \) as

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 - (1 - 2(1)^2)}{h} = \lim_{h \to 0} \frac{1 + h - 2(1 + 2h + h^2)}{h} = \lim_{h \to 0} \frac{-3h - 2h^2}{h} = \lim_{h \to 0} \frac{-3 - 2h}{h} = -3.
\]

Exercise 3.1.28

Exercise 3.1.28. Find an equation for the straight line having slope \( 1/4 \) that is tangent to the curve \( y = \sqrt{x} \).

Solution. We find the derivative of \( y = f(x) = \sqrt{x} \) at point \( x_0 \). The derivative gives the slope of the curve at the point \((x_0, f(x_0))\), so we’ll set the derivative equal to the desired slope \(1/4\) and determine \(x_0\) from the resulting equation. The derivative of \( y = f(x) = \sqrt{x} \) at point \( x_0 \) is

\[
f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} = \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})(\sqrt{x_0 + h} + \sqrt{x_0})}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})}{h} \cdot \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} \cdot \frac{1}{h} = \frac{1}{2\sqrt{x_0}}.
\]

Exercise 3.1.28 (continued)

Exercise 3.1.28. Find an equation for the straight line having slope \( 1/4 \) that is tangent to the curve \( y = \sqrt{x} \).

Solution (continued). ...
Exercise 3.1.30. Speed of a rocket. At t sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

Solution. The instantaneous velocity at time $t = t_0$ is

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h \to 0} \frac{3(t_0 + h)^2 - 3(t_0)^2}{h}$$

$$= \lim_{h \to 0} \frac{3t_0^2 + 6t_0h + 3h^2 - 3t_0^2}{h} = \lim_{h \to 0} \frac{6t_0h + 3h^2}{h} = \lim_{h \to 0} \frac{6t_0 + 3h}{1} = 6t_0 + 3h$$

So 10 sec after liftoff when $t_0 = 10$ sec, the rocket has velocity

$$f'(10) = 6(10) = 60 \text{ ft/sec}.$$ 

Exercise 3.1.42. Does the graph of $f(x) = x^{3/5}$ have a vertical tangent line at the origin?

Solution. First, notice that $f(0) = (0)^{3/5} = 0$ so that the graph of $y = f(x) = x^{3/5}$ does actually pass through the origin. We consider a limit of the difference quotient at $x_0 = 0$:

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^{3/5} - 0^{3/5}}{h} = \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}}.$$ 

Now $\lim_{h \to 0} 1 \neq 0$, $\lim_{h \to 0} h^{2/5} = 0$ (by the Root Rule, Theorem 2.1(7)), since $h^{2/5} = (h^{1/5})^2 = 1/(\sqrt[5]{h})^2 \geq 0$ for all $h$, so by Dr. Bob’s Infinite Limits Theorem we have $\lim_{h \to 0^{\pm}} 1/h^{2/5} = \pm \infty$. Since $1 > 0$ (D’uh!) and $h^{2/5} = (h^{1/5})^2 = (\sqrt[5]{h})^2 \geq 0$ for all $h$, then we have the “sign diagram” $1/h^{2/5} = (+)/(+) = +$. So $\lim_{h \to 0} 1/h^{2/5} = +\infty$, and the graph of $f(x) = x^{3/5}$ has a vertical tangent line at the origin. (Continued →)

Exercise 3.1.42 (continued)

Note. All this stuff with Dr. Bob’s Infinite Limits Theorem and a sign diagram is necessary! In Exercise 3.1.40 we address the existence of a vertical tangent of $y = f(x) = x^{4/5}$ at the origin. In this problem we find that $\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{1}{h^{1/5}}$. We find from the sign diagram that $\lim_{h \to 0^{\pm}} 1/h^{1/5} = -\infty$ and $\lim_{h \to 0^{\pm}} 1/h^{1/5} = -\infty$. So the two-sided limit does not exist and so the graph of $f(x) = x^{4/5}$ does not have a vertical tangent line at the origin. In fact the graph has a “cusp” at the origin:

![No Vertical Tangent Line at Origin](image)