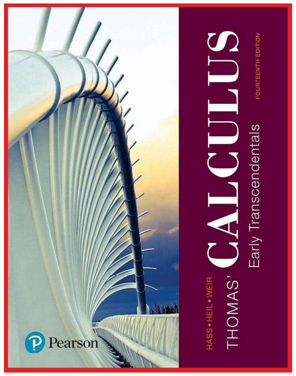
Calculus 1

Chapter 3. Derivatives

3.1. Tangent Lines and the Derivative at a Point—Examples and Proofs



Exercise 3.1.7

Exercise 3.1.7. Find an equation for the tangent line to the curve $y = 2\sqrt{x}$ at the point $(1, 2)$. Then sketch the curve and tangent line together.

Solution. With $y = f(x) = 2\sqrt{x}$ and $P(x_0, f(x_0)) = (1, 2)$, we have the slope of the curve $y = f(x)$ as

$$\begin{aligned} m &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim_{h\to 0}\frac{2\sqrt{1+h}-2}{h} \\ &= \lim_{h\to 0}\frac{2\sqrt{1+h}-2}{h}\left(\frac{2\sqrt{1+h}+2}{2\sqrt{1+h}+2}\right) \text{ multiplying by a form of 1} \\ &\quad\ \text{involving the conjugate} \\ &= \lim_{h\to 0}\frac{(2\sqrt{1+h}-2)(2\sqrt{1+h}+2)}{h(2\sqrt{1+h}+2)} = \lim_{h\to 0}\frac{(2\sqrt{1+h})^2-(2)^2}{h(2\sqrt{1+h}+2)} \\ &= \lim_{h\to 0}\frac{4(1+h)-4}{h(2\sqrt{1+h}+2)} = \lim_{h\to 0}\frac{4h}{h(2\sqrt{1+h}+2)} \end{aligned}$$

Exercise 3.1.7 (continued 1)

Solution (continued).

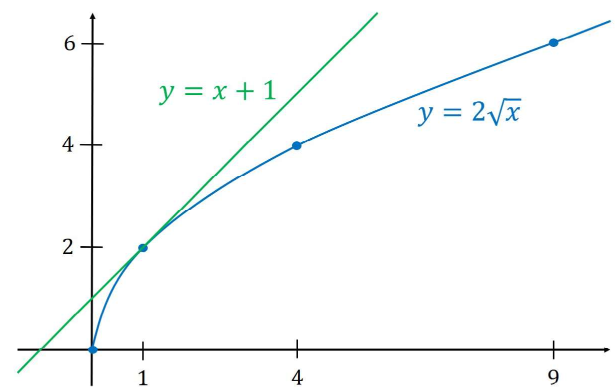
$$\begin{aligned} m &= \lim_{h\to 0}\frac{4h}{h(2\sqrt{1+h}+2)} \\ &= \lim_{h\to 0}\frac{4}{2\sqrt{1+h}+2} = \frac{4}{2\sqrt{1+(0)}+2} \text{ by the Sum Rule, Quotient} \\ &\quad\ \text{Rule, and Root Rule of Theorem 2.1, and Theorem 2.2} \\ &= \frac{4}{2\sqrt{1}+2} = 1. \end{aligned}$$

So the desired tangent line has slope $m = 1$ and passes through the point $(x_1, y_1) = (1, 2)$. By the point-slope formula, $y - y_1 = m(x - x_1)$, the tangent line is $y - (2) = (1)(x - (1))$ or $y - 2 = x - 1$ or $\boxed{y = x + 1}$.

Exercise 3.1.7 (continued 2)

Exercise 3.1.7. Find an equation for the tangent line to the curve $y = 2\sqrt{x}$ at the point $(1, 2)$. Then sketch the curve and tangent line together.

Solution (continued). The graphs of $y = 2\sqrt{x}$ and $y = x + 1$ are:



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Exercise 3.1.12.

Exercise 3.1.12. Find the slope of the graph of function $f(x) = x - 2x^2$ at the point $(1, -1)$. Then find an equation for the line tangent to the graph there.

Solution. With $y = f(x) = x - 2x^2$ and $P(x_0, f(x_0))) = (1, -1)$, we have the slope of the curve $y = f(x)$ as

$$\begin{aligned} m &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{((1+h) - 2(1+h)^2) - ((1) - 2(1)^2)}{h} \\ &= \lim_{h \to 0} \frac{1 + h - 2(1 + 2h + h^2) - (-1)}{h} \\ &= \lim_{h \to 0} \frac{1 + h - 2 - 4h - 2h^2 + 1}{h} = \lim_{h \to 0} \frac{-3h - 2h^2}{h} = \lim_{h \to 0} \frac{h(-3 - 2h)}{h} \\ &= \lim_{h \to 0} (-3 - 2h) = -3 - 2(0) = -3. \end{aligned}$$

Exercise 3.1.12 (continued).

Exercise 3.1.12. Find the slope of the graph of function $f(x) = x - 2x^2$ at the point $(1, -1)$. Then find an equation for the line tangent to the graph there.

Solution (continued). So the desired tangent line has slope $m = -3$ and passes through the point $(x_1, y_1) = (1, -1)$. By the point-slope formula, $y - y_1 = m(x - x_1)$, the tangent line is $y - (-1) = (-3)(x - (1))$ or $y + 1 = -3x + 3$ or $\boxed{y = -3x + 2}$. $\square$

Exercise 3.1.28

Exercise 3.1.28. Find an equation for the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Solution. We find the derivative of $y = f(x) = \sqrt{x}$ at point x_0. The derivative gives the slope of the curve at the point $(x_0, f(x_0))$, so we'll set the derivative equal to the desired slope $1/4$ and determine x_0 from the resulting equation. The derivative of $y = f(x) = \sqrt{x}$ at point x_0 is

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \left(\frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \right) \\ &= \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})(\sqrt{x_0 + h} + \sqrt{x_0})}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\ &= \lim_{h \to 0} \frac{(\sqrt{x_0 + h})^2 - (\sqrt{x_0})^2}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \end{aligned}$$

Exercise 3.1.28 (continued)

Exercise 3.1.28. Find an equation for the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.
Solution (continued). ...

$$\begin{aligned} f'(x_0) &= = \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\ &= \lim_{h \to 0} \frac{1}{(\sqrt{x_0 + h} + \sqrt{x_0})} = \frac{1}{(\sqrt{x_0 + 0} + \sqrt{x_0})} \text{ by the Sum Rule,} \\ &\quad \text{Quotient Rule, and Root Rule of Theorem 2.1} \\ &= \frac{1}{2\sqrt{x_0}}. \end{aligned}$$

So we set $1/4 = 1/(2\sqrt{x_0})$ to get $x_0 = 4$. So the desired tangent line has slope $m = 1/4$ and passes through the point $(x_0, f(x_0)) = (4, \sqrt{4}) = (4, 2) = (x_1, y_1)$. By the point-slope formula, $y - y_1 = m(x - x_1)$, the tangent line is $y - (2) = (1/4)(x - (4))$ or $y - 2 = x/4 - 1$ or $\boxed{y = x/4 + 1}$. $\square$

Exercise 3.1.30

Exercise 3.1.30. Speed of a rocket. At t sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

Solution. The instantaneous velocity at time $t = t_0$ is

$$\begin{aligned} f'(t_0) &= \lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} = \lim_{h\to 0} \frac{3(t_0+h)^2 - 3(t_0)^2}{h} \\ &= \lim_{h\to 0} \frac{3(t_0^2 + 2t_0 h + h^2) - 3t_0^2}{h} = \lim_{h\to 0} \frac{3t_0^2 + 6t_0 h + 3h^2 - 3t_0^2}{h} \\ &= \lim_{h\to 0} \frac{6t_0 h + 3h^2}{h} = \lim_{h\to 0} \frac{h(6t_0 + 3h)}{h} \\ &= \lim_{h\to 0} (6t_0 + 3h) = 6t_0 + 3(0) = 6t_0 \text{ ft/sec.} \end{aligned}$$

So 10 sec after liftoff when $t_0 = 10$ sec, the rocket has velocity $f'(10) = 6(10) = \boxed{60 \text{ ft/sec}}$. $\square$

Exercise 3.1.42

Exercise 3.1.42. Does the graph of $f(x) = x^{3/5}$ have a vertical tangent line at the origin?

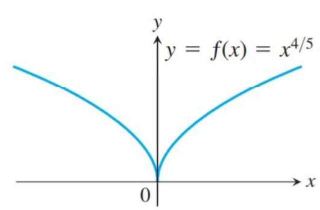
Solution. First, notice that $f(0) = (0)^{3/5} = 0$ so that the graph of $y = f(x) = x^{3/5}$ does actually pass through the origin. We consider a limit of the difference quotient at $x_0 = 0$:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^{3/5} - (0)^{3/5}}{h} = \lim_{h\to 0} \frac{h^{3/5}}{h} = \lim_{h\to 0} \frac{1}{h^{2/5}}.$$

Now $\lim_{h\to 0} 1 = 1 \neq 0$, $\lim_{h\to 0} h^{2/5} = 0$ (by the Root Rule, Theorem 2.1(7), since $h^{2/5} = (h^{1/5})^2 = 1/(\sqrt[5]{x})^2 \geq 0$ for all h), so by Dr. Bob's Infinite Limits Theorem we have $\lim_{h\to 0^\pm} 1/h^{2/5} = \pm\infty$. Since $1 > 0$ (D'uh!) and $h^{2/5} = (h^{1/5})^2 = (\sqrt[5]{h})^2 \geq 0$ for all h, then we have the "sign diagram": $1/h^{2/5} = (+)/(+) = +$. So $\lim_{h\to 0} 1/h^{2/5} = +\infty$, and $\boxed{\text{YES}}$ the graph of $f(x) = x^{3/5}$ has a vertical tangent line at the origin. (Continued $\longrightarrow$)

Exercise 3.1.42 (continued)

Note. All this stuff with Dr. Bob's Infinite Limits Theorem and a sign diagram is necessary! In Exercise 3.1.40 we address the existence of a vertical tangent of $y = f(x) = x^{4/5}$ at the origin. In this problem we find that $\displaystyle\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{1}{h^{1/5}}$. We find from the sign diagram that $\lim_{h\to 0^-} 1/h^{1/5} = -\infty$ and $\lim_{h\to 0^+} 1/h^{1/5} = \infty$. So the two-sided limit does not exist and so the graph of $f(x) = x^{4/5}$ does not have a vertical tangent line at the origin. In fact the graph has a "cusp" at the origin:



NO VERTICAL TANGENT LINE AT ORIGIN

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