Chapter 3. Derivatives
3.1. Tangent Lines and the Derivative at a Point—Examples and Proofs
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**Exercise 3.1.7.** Find an equation for the tangent line to the curve \( y = 2\sqrt{x} \) at the point \((1, 2)\). Then sketch the curve and tangent line together.

**Solution.** With \( y = f(x) = 2\sqrt{x} \) and \( P(x_0, f(x_0)) = (1, 2) \), we have the slope of the curve \( y = f(x) \) as

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2\sqrt{1}}{h} = \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{2\sqrt{1 + h} - 2}{h} \left( \frac{2\sqrt{1 + h} + 2}{2\sqrt{1 + h} + 2} \right) \text{ multiplying by a form of 1 involving the conjugate}
\]
Exercise 3.1.7

Exercise 3.1.7. Find an equation for the tangent line to the curve \( y = 2\sqrt{x} \) at the point \((1, 2)\). Then sketch the curve and tangent line together.

Solution. With \( y = f(x) = 2\sqrt{x} \) and \( P(x_0, f(x_0)) = (1, 2) \), we have the slope of the curve \( y = f(x) \) as

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= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h} \left( \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2} \right) \text{ multiplying by a form of } 1 \text{ involving the conjugate}
\]

\[
= \lim_{h \to 0} \frac{(2\sqrt{1+h} - 2)(2\sqrt{1+h} + 2)}{h(2\sqrt{1+h} + 2)} = \lim_{h \to 0} \frac{(2\sqrt{1+h})^2 - (2)^2}{h(2\sqrt{1+h} + 2)}
\]

\[
= \lim_{h \to 0} \frac{4(1+h) - 4}{h(2\sqrt{1+h} + 2)} = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1+h} + 2)}
\]
**Exercise 3.1.7**

**Exercise 3.1.7.** Find an equation for the tangent line to the curve $y = 2\sqrt{x}$ at the point $(1, 2)$. Then sketch the curve and tangent line together.

**Solution.** With $y = f(x) = 2\sqrt{x}$ and $P(x_0, f(x_0)) = (1, 2)$, we have the slope of the curve $y = f(x)$ as

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{2\sqrt{1+h} - 2\sqrt{1}}{h} = \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{2\sqrt{1+h} - 2}{h} \left( \frac{2\sqrt{1+h} + 2}{2\sqrt{1+h} + 2} \right) \text{ multiplying by a form of 1 involving the conjugate}$$

$$= \lim_{h \to 0} \frac{(2\sqrt{1+h} - 2)(2\sqrt{1+h} + 2)}{h(2\sqrt{1+h} + 2)} = \lim_{h \to 0} \frac{(2\sqrt{1+h})^2 - (2)^2}{h(2\sqrt{1+h} + 2)}$$

$$= \lim_{h \to 0} \frac{4(1+h) - 4}{h(2\sqrt{1+h} + 2)} = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1+h} + 2)}$$
**Exercise 3.1.7 (continued 1)**

**Solution (continued).**

\[ m = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1 + h + 2})} \]

\[ = \lim_{h \to 0} \frac{4}{2\sqrt{1 + h + 2}} = \frac{4}{2\sqrt{1 + (0) + 2}} = \frac{4}{2\sqrt{1 + 2}} = 1. \]

By the Sum Rule, Quotient Rule, and Root Rule of Theorem 2.1, and Theorem 2.2

So the desired tangent line has slope \( m = 1 \) and passes through the point \((x_1, y_1) = (1, 2)\). By the point-slope formula, \( y - y_1 = m(x - x_1) \), the tangent line is \( y - 2 = (1)(x - 1) \) or \( y - 2 = x - 1 \) or \( y = x + 1 \).
Solution (continued).

\[ m = \lim_{h \to 0} \frac{4h}{h(2\sqrt{1 + h + 2})} \]

\[ = \lim_{h \to 0} \frac{4}{2\sqrt{1 + h + 2}} = \frac{4}{2\sqrt{1 + (0) + 2}} \]

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So the desired tangent line has slope \( m = 1 \) and passes through the point \((x_1, y_1) = (1, 2)\). By the point-slope formula, \( y - y_1 = m(x - x_1) \), the tangent line is \( y - (2) = (1)(x - (1)) \) or \( y - 2 = x - 1 \) or \( y = x + 1 \).
Exercise 3.1.7 (continued 2)

Exercise 3.1.7. Find an equation for the tangent line to the curve \( y = 2\sqrt{x} \) at the point \((1, 2)\). Then sketch the curve and tangent line together.

Solution (continued). The graphs of \( y = 2\sqrt{x} \) and \( y = x + 1 \) are:
Exercise 3.1.12. Find the slope of the graph of function $f(x) = x - 2x^2$ at the point $(1, -1)$. Then find an equation for the line tangent to the graph there.

Solution. With $y = f(x) = x - 2x^2$ and $P(x_0, f(x_0)) = (1, -1)$, we have the slope of the curve $y = f(x)$ as

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{((1 + h) - 2(1 + h)^2) - ((1) - 2(1)^2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + h - 2(1 + 2h + h^2) - (-1)}{h}$$

$$= \lim_{h \to 0} \frac{1 + h - 2 - 4h - 2h^2 + 1}{h} = \lim_{h \to 0} \frac{-3h - 2h^2}{h} = \lim_{h \to 0} \frac{h(-3 - 2h)}{h}$$

$$= \lim_{h \to 0} (-3 - 2h) = -3 - 2(0) = -3.$$
Exercise 3.1.12. Find the slope of the graph of function \( f(x) = x - 2x^2 \) at the point \((1, -1)\). Then find an equation for the line tangent to the graph there.

Solution. With \( y = f(x) = x - 2x^2 \) and \( P(x_0, f(x_0)) = (1, -1) \), we have the slope of the curve \( y = f(x) \) as

\[
m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{((1 + h) - 2(1 + h)^2) - ((1) - 2(1)^2)}{h}
\]

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= \lim_{h \to 0} \frac{1 + h - 2(1 + 2h + h^2) - (-1)}{h}
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= \lim_{h \to 0} \frac{1 + h - 2 - 4h - 2h^2 + 1}{h} = \lim_{h \to 0} \frac{-3h - 2h^2}{h} = \lim_{h \to 0} \frac{h(-3 - 2h)}{h}
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= \lim_{h \to 0} (-3 - 2h) = -3 - 2(0) = -3.
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Exercise 3.1.12 (continued).

Exercise 3.1.12. Find the slope of the graph of function $f(x) = x - 2x^2$ at the point $(1, -1)$. Then find an equation for the line tangent to the graph there.

Solution (continued). So the desired tangent line has slope $m = -3$ and passes through the point $(x_1, y_1) = (1, -1)$. By the point-slope formula, $y - y_1 = m(x - x_1)$, the tangent line is $y - (-1) = (-3)(x - (1))$ or $y + 1 = -3x + 3$ or $y = -3x + 2$. □
Exercise 3.1.28

Exercise 3.1.28. Find an equation for the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Solution. We find the derivative of $y = f(x) = \sqrt{x}$ at point $x_0$. The derivative gives the slope of the curve at the point $(x_0, f(x_0))$, so we’ll set the derivative equal to the desired slope $1/4$ and determine $x_0$ from the resulting equation. The derivative of $y = f(x) = \sqrt{x}$ at point $x_0$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \left( \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \right)$$
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$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \left( \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}} \right)$$

$$= \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})(\sqrt{x_0 + h} + \sqrt{x_0})}{h(\sqrt{x_0 + h} + \sqrt{x_0})}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x_0 + h})^2 - (\sqrt{x_0})^2}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0 + h} + \sqrt{x_0})}.$$
Exercise 3.1.28

Exercise 3.1.28. Find an equation for the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Solution. We find the derivative of $y = f(x) = \sqrt{x}$ at point $x_0$. The derivative gives the slope of the curve at the point $(x_0, f(x_0))$, so we’ll set the derivative equal to the desired slope $1/4$ and determine $x_0$ from the resulting equation. The derivative of $y = f(x) = \sqrt{x}$ at point $x_0$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x_0 + h} - \sqrt{x_0}}{h} \cdot \frac{\sqrt{x_0 + h} + \sqrt{x_0}}{\sqrt{x_0 + h} + \sqrt{x_0}}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x_0 + h} - \sqrt{x_0})(\sqrt{x_0 + h} + \sqrt{x_0})}{h(\sqrt{x_0 + h} + \sqrt{x_0})}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x_0 + h})^2 - (\sqrt{x_0})^2}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0 + h} + \sqrt{x_0})}$$. 


Exercise 3.1.28 (continued)

**Exercise 3.1.28.** Find an equation for the straight line having slope 1/4 that is tangent to the curve \( y = \sqrt{x} \).

**Solution (continued).**

\[
\begin{align*}
 f'(x_0) &= \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0 + h} + \sqrt{x_0})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x_0 + h} + \sqrt{x_0})} \\
 &= \lim_{h \to 0} \frac{1}{\sqrt{x_0 + h} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0} + 0 + \sqrt{x_0}} \\
 &= \frac{1}{2\sqrt{x_0}}.
\end{align*}
\]

So we set \( 1/4 = 1/(2\sqrt{x_0}) \) to get \( x_0 = 4 \). So the desired tangent line has slope \( m = 1/4 \) and passes through the point \( (x_0, f(x_0)) = (4, \sqrt{4}) = (4, 2) = (x_1, y_1) \). By the point-slope formula, \( y - y_1 = m(x - x_1) \), the tangent line is \( y - 2 = (1/4)(x - 4) \) or \( y - 2 = x/4 - 1 \) or \( \boxed{y = x/4 + 1} \). \( \square \)
Exercise 3.1.28. Find an equation for the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

Solution (continued). . .

\[
\begin{align*}
\frac{f'(x_0)}{} &= \lim_{h \to 0} \frac{(x_0 + h) - (x_0)}{h(\sqrt{x_0} + h + \sqrt{x_0})} \\
&= \lim_{h \to 0} \frac{1}{h(\sqrt{x_0} + h + \sqrt{x_0})} = \lim_{h \to 0} \frac{1}{h(\sqrt{x_0} + 0 + \sqrt{x_0})} \\
&= \frac{1}{2\sqrt{x_0}}.
\end{align*}
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So we set $1/4 = 1/(2\sqrt{x_0})$ to get $x_0 = 4$. So the desired tangent line has slope $m = 1/4$ and passes through the point $(x_0, f(x_0)) = (4, \sqrt{4}) = (4, 2) = (x_1, y_1)$. By the point-slope formula, $y - y_1 = m(x - x_1)$, the tangent line is $y - (2) = (1/4)(x - (4))$ or $y - 2 = x/4 - 1$ or $y = x/4 + 1$. □
**Exercise 3.1.30. Speed of a rocket.** At $t$ sec after liftoff, the height of a rocket is $3t^2$ ft. How fast is the rocket climbing 10 sec after liftoff?

**Solution.** The instantaneous velocity at time $t = t_0$ is

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = \lim_{h \to 0} \frac{3(t_0 + h)^2 - 3(t_0)^2}{h}$$

$$= \lim_{h \to 0} \frac{3(t_0^2 + 2t_0h + h^2) - 3t_0^2}{h} = \lim_{h \to 0} \frac{3t_0^2 + 6t_0h + 3h^2 - 3t_0^2}{h}$$

$$= \lim_{h \to 0} \frac{6t_0h + 3h^2}{h} = \lim_{h \to 0} \frac{h(6t_0 + 3h)}{h}$$

$$= \lim_{h \to 0} (6t_0 + 3h) = 6t_0 + 3(0) = 6t_0 \text{ ft/sec}.$$
Exercise 3.1.30

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$$= \lim_{h \to 0} \frac{3(t_0^2 + 2t_0h + h^2) - 3t_0^2}{h} = \lim_{h \to 0} \frac{3t_0^2 + 6t_0h + 3h^2 - 3t_0^2}{h}$$

$$= \lim_{h \to 0} \frac{6t_0h + 3h^2}{h} = \lim_{h \to 0} \frac{h(6t_0 + 3h)}{h}$$

$$= \lim_{h \to 0} (6t_0 + 3h) = 6t_0 + 3(0) = 6t_0 \text{ ft/sec}.$$ 

So 10 sec after liftoff when $t_0 = 10$ sec, the rocket has velocity $f'(10) = 6(10) = 60 \text{ ft/sec}$. □
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$$= \lim_{h \to 0} \frac{3(t_0^2 + 2t_0h + h^2) - 3t_0^2}{h} = \lim_{h \to 0} \frac{3t_0^2 + 6t_0h + 3h^2 - 3t_0^2}{h}$$

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So 10 sec after liftoff when $t_0 = 10$ sec, the rocket has velocity $f'(10) = 6(10) = 60 \text{ ft/sec}$. □
Exercise 3.1.42

**Exercise 3.1.42.** Does the graph of \( f(x) = x^{3/5} \) have a vertical tangent line at the origin?

**Solution.** First, notice that \( f(0) = (0)^{3/5} = 0 \) so that the graph of \( y = f(x) = x^{3/5} \) does actually pass through the origin. We consider a limit of the difference quotient at \( x_0 = 0 \):

\[
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^{3/5} - (0)^{3/5}}{h} = \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}}.
\]
Exercise 3.1.42

Exercise 3.1.42. Does the graph of $f(x) = x^{3/5}$ have a vertical tangent line at the origin?

Solution. First, notice that $f(0) = (0)^{3/5} = 0$ so that the graph of $y = f(x) = x^{3/5}$ does actually pass through the origin. We consider a limit of the difference quotient at $x_0 = 0$:

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\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^{3/5} - (0)^{3/5}}{h} = \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}}.
$$

Now $\lim_{h \to 0} 1 = 1 \neq 0$, $\lim_{h \to 0} h^{2/5} = 0$ (by the Root Rule, Theorem 2.1(7), since $h^{2/5} = (h^{1/5})^2 = 1/(\sqrt[5]{h})^2 \geq 0$ for all $h$), so by Dr. Bob’s Infinite Limits Theorem we have $\lim_{h \to 0 \pm} 1/h^{2/5} = \pm \infty$. 
Exercise 3.1.42

Exercise 3.1.42. Does the graph of \( f(x) = x^{3/5} \) have a vertical tangent line at the origin?

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\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^{3/5} - (0)^{3/5}}{h} = \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}}.
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Now \( \lim_{h \to 0} 1 = 1 \neq 0 \), \( \lim_{h \to 0} h^{2/5} = 0 \) (by the Root Rule, Theorem 2.1(7), since \( h^{2/5} = (h^{1/5})^2 = \frac{1}{(\sqrt[5]{h})^2} \geq 0 \) for all \( h \)), so by Dr. Bob’s Infinite Limits Theorem we have \( \lim_{h \to 0} 1 / h^{2/5} = \pm \infty \). Since 1 > 0 (D’uh!) and \( h^{2/5} = (h^{1/5})^2 = (\sqrt[5]{h})^2 \geq 0 \) for all \( h \), then we have the “sign diagram”: \( 1 / h^{2/5} = (+)/(+) = + \). So \( \lim_{h \to 0} 1 / h^{2/5} = +\infty \), and **YES** the graph of \( f(x) = x^{3/5} \) has a vertical tangent line at the origin.

(Continued →)
Exercise 3.1.42. Does the graph of $f(x) = x^{3/5}$ have a vertical tangent line at the origin?

Solution. First, notice that $f(0) = (0)^{3/5} = 0$ so that the graph of $y = f(x) = x^{3/5}$ does actually pass through the origin. We consider a limit of the difference quotient at $x_0 = 0$:

$$
\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{(0 + h)^{3/5} - (0)^{3/5}}{h} = \lim_{h \to 0} \frac{h^{3/5}}{h} = \lim_{h \to 0} \frac{1}{h^{2/5}}.
$$

Now $\lim_{h \to 0} 1 = 1 \neq 0$, $\lim_{h \to 0} h^{2/5} = 0$ (by the Root Rule, Theorem 2.1(7), since $h^{2/5} = (h^{1/5})^2 = 1/(\sqrt[5]{x})^2 \geq 0$ for all $h$), so by Dr. Bob’s Infinite Limits Theorem we have $\lim_{h \to 0 \pm} 1/h^{2/5} = \pm\infty$. Since $1 > 0$ (D’uh!) and $h^{2/5} = (h^{1/5})^2 = (\sqrt[5]{h})^2 \geq 0$ for all $h$, then we have the “sign diagram”: $1/h^{2/5} = (+)/(+) = +$. So $\lim_{h \to 0} 1/h^{2/5} = +\infty$, and YES the graph of $f(x) = x^{3/5}$ has a vertical tangent line at the origin. (Continued →)
Note. All this stuff with Dr. Bob’s Infinite Limits Theorem and a sign diagram is necessary! In Exercise 3.1.40 we address the existence of a vertical tangent of $y = f(x) = x^{4/5}$ at the origin. In this problem we find that $\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{1}{h^{1/5}}$. We find from the sign diagram that $\lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty$ and $\lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty$. So the two-sided limit does not exist and so the graph of $f(x) = x^{4/5}$ does not have a vertical tangent line at the origin. In fact the graph has a “cusp” at the origin:
Note. All this stuff with Dr. Bob’s Infinite Limits Theorem and a sign diagram is necessary! In Exercise 3.1.40 we address the existence of a vertical tangent of $y = f(x) = x^{4/5}$ at the origin. In this problem we find that

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{1}{h^{1/5}}.$$ 

We find from the sign diagram that

$$\lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty \quad \text{and} \quad \lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty.$$ 

So the two-sided limit does not exist and so the graph of $f(x) = x^{4/5}$ does not have a vertical tangent line at the origin. In fact the graph has a “cusp” at the origin:
Exercise 3.1.42 (continued)

**Note.** All this stuff with Dr. Bob’s Infinite Limits Theorem and a sign diagram is necessary! In Exercise 3.1.40 we address the existence of a vertical tangent of $y = f(x) = x^{4/5}$ at the origin. In this problem we find that

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{1}{h^{1/5}}.$$  

We find from the sign diagram that

$$\lim_{h \to 0^-} \frac{1}{h^{1/5}} = -\infty \text{ and } \lim_{h \to 0^+} \frac{1}{h^{1/5}} = \infty.$$  

So the two-sided limit does not exist and so the graph of $f(x) = x^{4/5}$ does not have a vertical tangent line at the origin. In fact the graph has a “cusp” at the origin:

![Graph of $f(x) = x^{4/5}$](image-url)