Exercise 3.2.10

Exercise 3.2.10. Find the derivative $\frac{dv}{dt}$ where $v = t - \frac{1}{t}$.

Solution. By the definition of derivative we have

$$\frac{dv}{dt} = \lim_{h \to 0} \frac{v(t + h) - v(t)}{h} = \lim_{h \to 0} \frac{(t + h) - \frac{1}{t + h} - (t - \frac{1}{t})}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left( h + \left( \frac{-1}{t + h} + \frac{1}{t} \right) \right) = \lim_{h \to 0} \frac{1}{h} \left( h + \frac{-t}{t(t + h)} + \frac{t + h}{t(t + h)} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left( 1 + \frac{1}{t(t + h)} \right) = 1 + \frac{1}{t(t + 0)} = 1 + \frac{1}{t^2} \quad \Box$$

Example 3.2.3

Example 3.2.3. Consider the graphs of $y = f(x)$ and $y = f'(x)$:

At point $A$ the slope of $f$ is 0, so at point $A'$ (with the same $x$-value as point $A$) the value of $f'$ is 0. At point $B$ the slope of $f$ is $-1$, so at point $B'$ the value of $f'$ is $-1$. At point $C$ the slope of $f$ is $-4/3$, so at point $C'$ the value of $f'$ is $-4/3$. At point $D$ the slope of $f$ is 0, so at point $D'$ the value of $f'$ is 0. At point $E$ the slope of $f$ is $\approx 2$, so at point $E'$ the value of $f'$ is $\approx 2$.

Notice that when $f$ is decreasing (which happens between points $A$ and $D$) that $f'$ is negative. When $f$ is increasing (which happens to the right of point $D$) then $f'$ is positive. When the graph of $f$ "levels off" (which happens at points $A$ and $D$) then $f'$ has an $x$-intercept. $\Box$
Exercise 3.2.14

Exercise 3.2.14. Differentiate the function \( k(x) = \frac{1}{2+x} \) and find the slope of the tangent line at the value \( x = 2 \).

Solution. We have

\[
k'(x) = \lim_{h \to 0} \frac{k(x + h) - k(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{2+(x+h)} - \frac{1}{2+x}}{h}
= \lim_{h \to 0} \frac{1}{h} \left( \frac{2 + x}{(2+x)(2+x+h)} - \frac{2 + x + h}{(2+x)(2+x+h)} \right)
= \lim_{h \to 0} \frac{1}{h} \frac{-h}{(2+x)(2+x+h)} = \lim_{h \to 0} \frac{-1}{(2+x)(2+x+h)}
= \frac{-1}{(2+x)(2+x+0)} = \frac{-1}{(2+2)^2} = \frac{-1}{16} \quad \square
\]

Now the slope of \( k(x) \) at \( x = 2 \) is \( m = k'(2) = \frac{-1}{(2+2)^2} = \frac{-1}{16} \quad \square \]

Exercise 3.2.24

Exercise 3.2.24. An alternative formula for the derivative of \( f \) at \( x \) is

\[
f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.
\]

Use this formula to find the derivative of \( f(x) = x^2 - 3x + 4 \).

Solution. By the alternative formula we have

\[
f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{(z^2 - 3z + 4) - (x^2 - 3x + 4)}{z - x}
= \lim_{z \to x} \frac{(z^2 - x^2) - 3(z - x)}{z - x} = \lim_{z \to x} \frac{(z - x)(z + x) - 3(z - x)}{z - x}
= \lim_{z \to x} (z + x) - 3 = ((x) + x) - 3 = 2x - 3.
\]

\( \square \)

Exercise 3.2.30

Exercise 3.2.30. Match the given function with the derivative graphed in figures (a)–(d).

\[
\text{Solution. Since } y = f_4(x) \text{ has horizontal tangents at three points, then the graph of } y = f'_4(x) \text{ must have three } x \text{-intercepts. So the derivative must be graphed in (c).}
\]

Exercise 3.2.30 (continued)

Solution (continued). Notice that the graph of \( y = f_4(x) \) is decreasing until it reaches a minimum (indicated by the left-most blue arrow) and that the graph of \( y' \) is negative over the corresponding \( x \) values (where the intercept indicated by the left-most red arrow corresponds to this minimum of \( f_4 \)). The graph of \( y = f_4(x) \) is increasing until it reaches a maximum (indicated by the center blue arrow) and that the graph of \( y' \) is positive over the corresponding \( x \) values (where the intercept indicated by the center red arrow corresponds to this maximum of \( f_4 \)). Next, the graph of \( y = f_4(x) \) is decreasing between the origin and the right-most blue arrow and the graph of \( y' \) is negative over the corresponding \( x \) values (between the origin and the right-most red arrow). Finally, the graph of \( y = f_4(x) \) is increasing to the right of the right-most blue arrow and the graph of \( y' \) is positive over the corresponding \( x \) values (to the right of the right-most red arrow). \( \square \)
Exercise 3.2.44

Exercise 3.2.44. Determine if the piecewise defined function \( g \) is differentiable at the origin:

\[
g(x) = \begin{cases} 
2x - x^3 - 1, & x \geq 0 \\
\frac{x}{1-x^2}, & x < 0
\end{cases}
\]

Solution. Since \( g \) is piecewise defined, we consider left- and right-hand derivatives at 0. First, the right-hand derivative at 0 is:

\[
\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h}
\]

\[
= \lim_{h \to 0^+} \frac{(2(0+h) - (0+h)^3 - 1) - (2(0) - (0)^3 - 1)}{h} \quad \text{since } 0 + h > 0,
\]

we use the \( 2x - x^3 - 1 \) part of \( g \)

\[
= \lim_{h \to 0^+} \frac{2h - h^3 - 1 + 1}{h} = \lim_{h \to 0^+} \frac{2h - h^3}{h} = \frac{2 - h^2}{h}
\]

Exercise 3.2.44 (continued 1)

Solution (continued). . .

\[
= \lim_{h \to 0^+} \frac{2h - h^3 - 1 + 1}{h} = \lim_{h \to 0^+} \frac{h(2 - h^2)}{h} 
\]

\[
= \lim_{h \to 0^+} (2 - h^2) = 2 - (0)^2 = 2.
\]

Next, the left-hand derivative at 0 is:

\[
\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h}
\]

\[
= \lim_{h \to 0^-} \frac{((0+h) - \frac{1}{(0+h)+1}) - (2(0) - (0)^3 - 1)}{h} \quad \text{since } 0 + h < 0,
\]

we use the \( x - \frac{1}{x+1} \) part of \( g \)

\[
= \lim_{h \to 0^-} \frac{1}{h} \left( \left( h - \frac{1}{h+1} \right) + 1 \right) = \lim_{h \to 0^-} \frac{1}{h} \left( h + 1 + (h+1) \right)
\]

Exercise 3.2.44 (continued 2)

Solution (continued). . .

\[
= \lim_{h \to 0^-} \frac{1}{h} \left( \frac{h(h+1) - 1 + (h+1)}{h+1} \right) = \lim_{h \to 0^-} \frac{1}{h} \left( \frac{h^2 + h - 1 + h + 1}{h+1} \right)
\]

\[
= \lim_{h \to 0^-} \frac{1}{h} \left( h^2 + 2h \right) = \lim_{h \to 0^-} \frac{1}{h} \left( h \right) \quad \text{since } h + 2 = 0 + 2 = 2,
\]

\[
= \lim_{h \to 0^-} \frac{h + 1}{h} = 0 + 1 = 2.
\]

Since the left- and right-hand derivatives exist and are equal, then by Theorem 2.6, “Relation Between One-Sided and Two-Sided Limits,” the (two-sided) derivative exists and is 2. □

Theorem 3.1

Theorem 3.1. Differentiability Implies Continuity

If \( f \) has a derivative at \( x = c \), then \( f \) is continuous at \( x = c \).

Proof. By the definition of continuity, we need to show that

\[
\lim_{{x \to c}} f(x) = f(c), \quad \text{or equivalently (see Exercise 2.5.71) that}
\]

\[
\lim_{{h \to 0}} f(c + h) = f(c).
\]

Then

\[
\lim_{{h \to 0}} f(c + h) = \lim_{{h \to 0}} \left( f(c) + \frac{f(c + h) - f(c)}{h} \right)
\]

\[
= \lim_{{h \to 0}} f(c) + \lim_{{h \to 0}} \frac{f(c + h) - f(c)}{h} \lim_{{h \to 0}} h
\]

\[
= f(c) + f'(c)(0)
\]

\[
= f(c).
\]

Therefore \( f \) is continuous at \( x = c \). □
Exercise 3.2.50

Exercise 3.2.50. Consider function \( f \) with domain \( D = [-3, 3] \) graphed below. At what domain points does the function appear to be (a) differentiable, (b) continuous but not differentiable, (c) neither continuous nor differentiable?

![Graph of f(x)](image)

**Solution.** (a) The graph indicates that \( f \) has a right-hand derivative at \(-3\) and a left-hand derivative at \(3\). The graph is "smooth" for all other \( x \in (-3, 3) \), except for \( x = \pm 2 \) where the graph has a corner. So \( f \) is differentiable on \([-3, -2) \cup (-2, 2) \cup (2, 3]\). □

Exercise 3.2.56

Exercise 3.2.56. Does any tangent line to the curve \( y = \sqrt{x} \) cross the \( x \)-axis at \( x = -1 \)? If so, find an equation for the line and the point of tangency. If not, why not?

**Solution.** First, a line with slope \( m \) which has \( x \) intercept \( x_1 = -1 \) is of the form \( y = m(x - x_1) = m(x - (-1)) = m(x + 1) \) by the slope-intercept form of a line. Now the derivative of \( y = f(x) = \sqrt{x} \) is

\[
\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.
\]

So the slope of \( y = f(x) = \sqrt{x} \) at \( x = x_0 \) is \( f'(x_0) = \frac{1}{2\sqrt{x_0}} \).

We now need \( x_0 \) such that \( y = m(x + 1) = \frac{1}{2\sqrt{x_0}} (x + 1) \) and we need this line to contain the point \((x_0, \sqrt{x_0})\). So we must have \( (x_0, \sqrt{x_0}) = (x_0, y_0) = \left( x_0, \frac{1}{2\sqrt{x_0}} (x_0 + 1) \right) \), or \( \sqrt{x_0} = \frac{1}{2\sqrt{x_0}} (x_0 + 1) \) or \( 2(\sqrt{x_0})^2 = x_0 + 1 \) and \(2x_0 = x_0 + 1 \) (where \( x_0 \geq 0 \) or \( x_0 = 1 \). When \( x_0 = 1 \) then \( y_0 = 1 \) and \( m = 1/(2\sqrt{1}) = 1/2 \). Therefore, \( y = (1/2)(x + 1) \) and the point of tangency to \( y = \sqrt{x} \) is \((x_0, y_0) = (1, 1)\). □