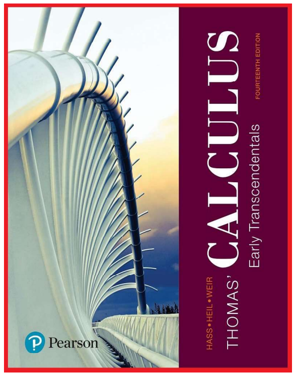
Calculus 1

Chapter 3. Derivatives

3.2. The Derivative as a Function—Examples and Proofs



Exercise 3.2.10

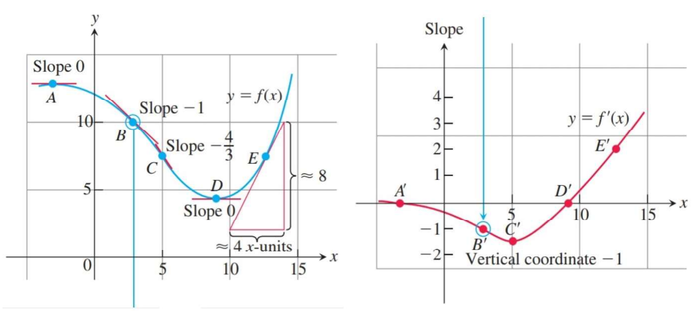
Exercise 3.2.10. Find the derivative $\dfrac{dv}{dt}$ where $v = t - \dfrac{1}{t}$.

Solution. By the definition of derivative we have

$$\begin{aligned}
\frac{dv}{dt} &= \lim_{h\to 0}\frac{v(t+h)-v(t)}{h} = \lim_{h\to 0}\frac{\left((t+h)-\frac{1}{(t+h)}\right)-\left(t-\frac{1}{t}\right)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left(h+\left(\frac{-1}{t+h}+\frac{1}{t}\right)\right) = \lim_{h\to 0}\frac{1}{h}\left(h+\frac{-t}{t(t+h)}+\frac{t+h}{t(t+h)}\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(h+\frac{h}{t(t+h)}\right) = \lim_{h\to 0}\frac{1}{h}\left(h\left(1+\frac{1}{t(t+h)}\right)\right)\\
&= \lim_{h\to 0}\left(1+\frac{1}{t(t+h)}\right) = 1+\frac{1}{t(t+(0))} = \boxed{1+\frac{1}{t^2}}. \qquad \square
\end{aligned}$$

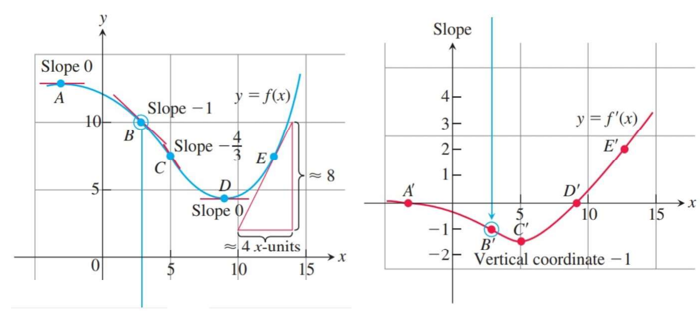
Example 3.2.3

Example 3.2.3. Consider the graphs of $y = f(x)$ and $y = f'(x)$:



At point A the slope of f is 0, so at point A' (with the same x-value as point A) the value of f' is 0. At point B the slope of f is -1, so at point B' the value of f' is -1. At point C the slope of f is $-4/3$, so at point C' the value of f' is $-4/3$. At point D the slope of f is 0, so at point D' the value of f' is 0. At point E the slope of f is ≈ 2, so at point E' the value of f' is ≈ 2.

Example 3.2.3 (continued)



Notice that when f is decreasing (which happens between points A and D) that f' is negative. When f is increasing (which happens to the right of point D) then f' is positive. When the graph of f "levels off" (which happens at points A and D) then f' has an x-intercept. $\square$

Exercise 3.2.14

Exercise 3.2.14. Differentiate the function $k(x) = \dfrac{1}{2+x}$ and find the slope of the tangent line at the value $x = 2$.

Solution. We have

$$\begin{aligned} k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+(x+h)} - \frac{1}{2+x}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{2+x}{(2+x)(2+x+h)} - \frac{2+x+h}{(2+x)(2+x+h)}\right) \\ &= \lim_{h\to 0} \frac{1}{h}\frac{-h}{(2+x)(2+x+h)} = \lim_{h\to 0}\frac{-1}{(2+x)(2+x+h)} \\ &= \frac{-1}{(2+x)(2+x+(0))} = \boxed{\frac{-1}{(2+x)^2}}. \end{aligned}$$

Now the slope of $k(x)$ at $x = 2$ is $m = k'(2) = \dfrac{-1}{(2+(2))^2} = \boxed{\dfrac{-1}{16}}$. □

Exercise 3.2.24

Exercise 3.2.24. An alternative formula for the derivative of f at x is

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}.$$

Use this formula to find the derivative of $f(x) = x^2 - 3x + 4$.

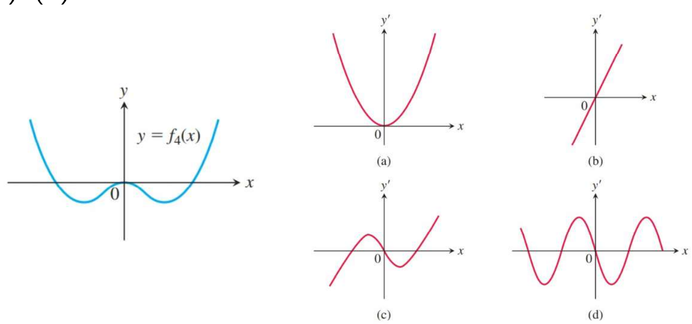
Solution. By the alternative formula we have

$$\begin{aligned} f'(x) &= \lim_{z\to x}\frac{f(z)-f(x)}{z-x} = \lim_{z\to x}\frac{(z^2-3z+4)-(x^2-3x+4)}{z-x} \\ &= \lim_{z\to x}\frac{(z^2-x^2)-3(z-x)}{z-x} = \lim_{z\to x}\frac{(z-x)(z+x)-3(z-x)}{z-x} \\ &= \lim_{z\to x}(z+x)-3 = ((x)+x)-3 = \boxed{2x-3}. \end{aligned}$$

□

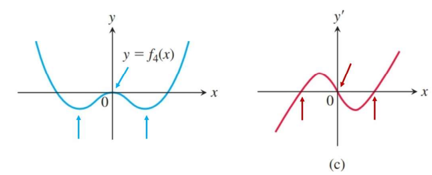
Exercise 3.2.30

Exercise 3.2.30. Match the given function with the derivative graphed in figures (a)–(d).



Solution. Since $y = f_4(x)$ has horizontal tangents at three points, then the graph of $y = f_4'(x)$ must have three x-intercepts. So the derivative must be graphed in (c).

Exercise 3.2.30 (continued)



Solution (continued). Notice that the graph of $y = f_4(x)$ is decreasing until it reaches a minimum (indicated by the left-most blue arrow) and that the graph of y' is negative over the corresponding x values (where the intercept indicated by the left-most red arrow corresponds to this minimum of f_4). The graph of $y = f_4(x)$ is increasing until it reaches a maximum (indicated by the center blue arrow) and that the graph of y' is positive over the corresponding x values (where the intercept indicated by the center red arrow corresponds to this maximum of f_4). Next, the graph of $y = f_4(x)$ is decreasing between the origin and the right-most blue arrow and the graph of y' is negative over the corresponding x values (between the origin and the right-most red arrow). Finally, the graph of $y = f_4(x)$ is increasing to the right of the right-most blue arrow and the graph of y' is positive over the corresponding x values (to the right of the right-most red arrow). □

Exercise 3.2.44

Exercise 3.2.44. Determine if the piecewise defined function g is differentiable at the origin:

$$g(x) = \begin{cases} 2x - x^3 - 1, & x \geq 0 \\ x - \frac{1}{x+1}, & x < 0 \end{cases}$$

Solution. Since g is piecewise defined, we consider left- and right-hand derivatives at 0. First, the right-hand derivative at 0 is:

$$\lim_{h \to 0^+} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \to 0^+} \frac{(2(0+h) - (0+h)^3 - 1) - (2(0) - (0)^3 - 1)}{h} \text{ since } 0 + h > 0,$$

we use the $2x - x^3 - 1$ part of g

$$= \lim_{h \to 0^+} \frac{2h - h^3 - 1 + 1}{h} = \lim_{h \to 0^+} \frac{h(2 - h^2)}{h}$$

Exercise 3.2.44 (continued 1)

Solution (continued). ...

$$= \lim_{h \to 0^+} \frac{2h - h^3 - 1 + 1}{h} = \lim_{h \to 0^+} \frac{h(2 - h^2)}{h}$$

$$= \lim_{h \to 0^+} (2 - h^2) = 2 - (0)^2 = 2.$$

Next, the left-hand derivative at 0 is:

$$\lim_{h \to 0^-} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \to 0^-} \frac{\left((0+h) - \frac{1}{(0+h)+1}\right) - (2(0) - (0)^3 - 1)}{h} \text{ since } 0 + h < 0,$$

we use the $x - \frac{1}{x+1}$ part of g

$$= \lim_{h \to 0^-} \frac{1}{h}\left(\left(h - \frac{1}{h+1}\right) + (1)\right) = \lim_{h \to 0^-} \frac{1}{h}\left(\frac{h(h+1) - 1 + (h+1)}{h+1}\right)$$

Exercise 3.2.44 (continued 2)

Solution (continued). ...

$$= \lim_{h \to 0^-} \frac{1}{h}\left(\frac{h(h+1) - 1 + (h+1)}{h+1}\right) = \lim_{h \to 0^-} \frac{1}{h}\left(\frac{h^2 + h - 1 + h + 1}{h+1}\right)$$

$$= \lim_{h \to 0^-} \frac{1}{h}\frac{h^2 + 2h}{h+1} = \lim_{h \to 0^-} \frac{1}{h}\frac{h(h+2)}{h+1}$$

$$= \lim_{h \to 0^-} \frac{h+2}{h+1} = \frac{(0)+2}{(0)+1} = 2.$$

Since the left- and right-hand derivatives exist and are equal, then by Theorem 2.6, "Relation Between One-Sided and Two-Sided Limits," the (two-sided) $\boxed{\text{derivative exists and is 2}}$. $\square$

Theorem 3.1

Theorem 3.1. Differentiability Implies Continuity
If f has a derivative at $x = c$, then f is continuous at $x = c$.

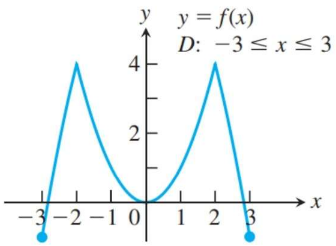
Proof. By the definition of continuity, we need to show that $\lim_{x \to c} f(x) = f(c)$, or equivalently (see Exercise 2.5.71) that $\lim_{h \to 0} f(c+h) = f(c)$. Then

$$\begin{aligned} \lim_{h \to 0} f(c+h) &= \lim_{h \to 0}\left(f(c) + \frac{f(c+h) - f(c)}{h} h\right) \\ &= \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \lim_{h \to 0} h \\ &= f(c) + f'(c)(0) \\ &= f(c). \end{aligned}$$

Therefore f is continuous at $x = c$. $\square$

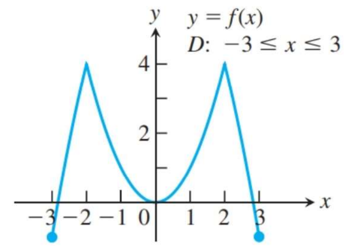
Exercise 3.2.50

Exercise 3.2.50. Consider function f with domain $D = [-3, 3]$ graphed below. At what domain points does the function appear to be **(a)** differentiable, **(b)** continuous but not differentiable, **(c)** neither continuous nor differentiable?



Solution. **(a)** The graph indicates that f has a right-hand derivative at -3 and a left-hand derivative at 3. The graph is "smooth" for all other $x \in (-3, 3)$, except for $x = \pm 2$ where the graph has a corner. So f is differentiable on $\boxed{[-3, -2) \cup (-2, 2) \cup (2, 3]}$. □

Exercise 3.2.50 (continued 1)



Solution. **(b)** The graph indicates that f is continuous on $[-3, 3]$ (by Dr. Bob's anthropomorphic idea of continuity, if you like). So f is $\boxed{\text{continuous but not differentiable at } \pm 2}$. □

(c) There are $\boxed{\text{no points where } f \text{ is neither continuous nor differentiable}}$ since f is continuous on $[-3, 3]$. □

Exercise 3.2.56

Exercise 3.2.56. Does any tangent line to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

Solution. First, a line with slope m which has x intercept $x_1 = -1$ is of the form $y = m(x - x_1) = m(x - (-1)) = m(x + 1)$ by the slope-intercept form of a line. Now the derivative of $y = f(x) = \sqrt{x}$ is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h\to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{(x+h) - (x)}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+(0)} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

Exercise 3.2.56 (continued)

Exercise 3.2.56. Does any tangent line to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

Solution. So the slope of $y = f(x) = \sqrt{x}$ at $x = x_0$ is $f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$.

We now need x_0 such that $y = m(x+1) = \dfrac{1}{2\sqrt{x_0}}(x+1)$ and we need this line to contain the point $(x_0, \sqrt{x_0})$. So we must have $(x_0, \sqrt{x_0}) = (x_0, y_0) = \left(x_0, \dfrac{1}{2\sqrt{x_0}}(x_0 + 1) \right)$, or $\sqrt{x_0} = \dfrac{1}{2\sqrt{x_0}}(x_0 + 1)$ or $2(\sqrt{x_0})^2 = x_0 + 1$ or $2x_0 = x_0 + 1$ (where $x_0 \geq 0$) or $x_0 = 1$. When $x_0 = 1$ then $y_0 = 1$ and $m = 1/(2\sqrt{1}) = 1/2$. Therefore $\boxed{\text{yes, there is such a line}}$, $\boxed{\text{it has equation } y = (1/2)(x+1)}$ and the $\boxed{\text{point of tangency to } y = \sqrt{x} \text{ is } (x_0, y_0) = (1, 1)}$. □