Example 3.7.A

Example 3.7.A. Find the slope of the line tangent to \( x^2 + y^2 = 1 \) at \((x, y) = (\sqrt{2}/2, \sqrt{2}/2)\). Do the same for the point \((x, y) = (\sqrt{2}/2, -\sqrt{2}/2)\).

Solution. We have just seen that implicit differentiation gives \( \frac{dy}{dx} = -\frac{x}{y} \). So at \((\sqrt{2}/2, \sqrt{2}/2)\), the slope of a line tangent to \( x^2 + y^2 = 1 \) is

\[
\left. \frac{dy}{dx} \right|_{(x,y)=(\sqrt{2}/2,\sqrt{2}/2)} = -\frac{(\sqrt{2}/2)}{(\sqrt{2}/2)} = -1
\]

At \((\sqrt{2}/2, -\sqrt{2}/2)\), the slope of a line tangent to \( x^2 + y^2 = 1 \) is

\[
\left. \frac{dy}{dx} \right|_{(x,y)=(\sqrt{2}/2,-\sqrt{2}/2)} = -\frac{(\sqrt{2}/2)}{(-\sqrt{2}/2)} = 1.
\]

Example 3.7.A (continued)

Solution (continued).

Since \( x^2 + y^2 = 1 \) does not determine a single function (it fails the vertical line test), then to find the slope of a tangent to the graph of the equation, we need both an \( x \) value and a \( y \) for the point of tangency.

Exercise 3.7.16

Exercise 3.7.16. Find \( \frac{dy}{dx} \) for \( y \) an implicit function of \( x \) given by the equation \( e^{x^2y} = 2x + 2y \).

Solution. Differentiating implicitly, \( \frac{d}{dx}[e^{x^2y}] = \frac{d}{dx}[2x + 2y] \) or

\[
e^{x^2y}[2x][y] + (x^2)[dy/dx] = 2[1] + 2[dy/dx] \quad \text{(notice that since \( y \) is a function of \( x \) then we must use the Derivative Product Rule to differentiate \( x^2y \)).}
\]

Solving for \( dy/dx \) we get \( e^{x^2y}(2xy + x^2)(dy/dx) = 2 + 2(dy/dx) \) or \( 2xye^{x^2y} + x^2e^{x^2y}(dy/dx) = 2 + 2(dy/dx) \) or

\[
x^2e^{x^2y}(dy/dx) - 2(dy/dx) = -2xye^{x^2y} + 2 \quad \text{or} \quad \left( \frac{dy}{dx} \right) = \frac{-2xye^{x^2y} + 2}{x^2e^{x^2y} - 2} \quad \blacksquare
\]
Exercise 3.7.20. Find $dr/d\theta$ for $r$ an implicit function of $\theta$ given by the equation $\cos r + \cot \theta = e^{r\theta}$.

Solution. Differentiating implicitly, $\frac{d}{d\theta} \left[ \cos r + \cot \theta \right] = \frac{d}{d\theta} [e^{r\theta}]$ or

$$-\sin r \frac{dr}{d\theta} = -e^{r\theta} \frac{dr}{d\theta} [\cos r][\frac{d}{d\theta} \theta] \left[ 1 \right]$$

(notice that since $r$ is a function of $\theta$ then we must use the Derivative Product Rule to differentiate $r\theta$) or $-\sin r \frac{dr}{d\theta} - \csc^2 \theta = e^{r\theta} \left[ \frac{dr}{d\theta} \right][\theta] + (r)[1]$ Solving for $dr/d\theta$ we get $-\sin r - \theta e^{r\theta} \frac{dr}{d\theta} = re^{r\theta} + \csc^2 \theta$ or $\frac{dr}{d\theta} = \frac{re^{r\theta} + \csc^2 \theta}{-\sin r - \theta e^{r\theta}}$. \qed

Exercise 3.7.40. Verify that the point $(\pi/4, \pi/2)$ is on the curve $x \sin 2y = y \cos 2x$ and find the equations of the lines that are (a) tangent and (b) normal to the curve at $(\pi/4, \pi/2)$.

Solution. First, with $(x, y) = (\pi/4, \pi/2)$ the equation $x \sin 2y = y \cos 2x$ becomes $\left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{2} \right) \frac{2}{2} \cos \left( \frac{\pi}{4} \right)$ or $\left( \frac{\pi}{4} \right) \sin \frac{\pi}{2} \left( \frac{\pi}{2} \right) \cos \left( \frac{\pi}{2} \right) \left( 0 \right) \frac{1}{2}$ $(0)$ or $0 \frac{1}{2}$, which is true and so $(\pi/4, \pi/2)$ is on the curve $x \sin 2y = y \cos 2x$.

(a) We differentiate the equation implicitly to find $dy/dx$ and we get

$$\frac{dy}{dx} \left[ x \sin 2y \right] = \frac{d}{dx} [y \cos 2x]$$

or $\left[ 1 \right](\sin 2y) + (x)[\cos(2y)](2dy/dx) = [dy/dx](\cos 2x) + (y)[-(\sin(2x)]$.

or $\sin 2y + 2x(\cos(2y))(2dy/dx) = (dy/dx)(\cos 2x) + (y)(-\sin(2x))$ or $2y + 2x(\cos(2y))(dy/dx) = (\cos 2x)(dy/dx) - 2y \sin 2x$.

Exercise 3.7.40 (continued 1)

Exercise 3.7.40 (continued 2)
Exercise 3.7.44. Normals Parallel to a Line.
Find the normals to the curve \( xy + 2x - y = 0 \) that are parallel to the line \( 2x + y = 0 \).

**Solution.** First we find \( dy/dx \) by differentiating implicitly to get:
\[
\frac{d}{dx}[xy + 2x - y] = \frac{d}{dx}[0] \quad \text{or} \quad (x)(dy/dx) + \frac{d}{dx}[2] - [dy/dx] = 0 \quad \text{or} \quad y + 2 + (x-1)dy/dx = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{-y - 2}{x - 1} = \frac{y + 2}{1 - x}.
\]
So at a point \((x_1, y_1)\) on the curve \( xy + 2x - y = 0 \), the slope of a normal line is
\[
-1 \left( \frac{dy}{dx} \right) \bigg|_{(x_1,y_1)} = -\frac{1 - x_1}{y_1 + 2}.
\]
Now the slope of line \( 2x + y = 0 \) is
\[
m = \frac{-2}{1} = -2 \quad \text{or} \quad y = -2x - 3.
\]
So we look for a point \((x_1, y_1)\) such that
\[
\frac{1 - x_1}{y_1 + 2} = -2 \quad \text{or} \quad 1 - x_1 = 2(y_1 + 2) = 2y_1 + 4 \quad \text{or} \quad x_1 = -2y_1 - 3.
\]

Exercise 3.7.22. Find \( dy/dx \) and \( d^2y/dx^2 \) for \( y \) an implicit function of \( x \) given by the equation \( x^{2/3} + y^{2/3} = 1 \).

**Solution.** We differentiate implicitly to get
\[
\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \left( \frac{dy}{dx} \right) = 0 \quad \text{or} \quad \frac{2}{3} y^{-1/3} \left( \frac{dy}{dx} \right) = -\frac{2}{3} x^{-1/3} \quad \text{or} \quad \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{x^{1/3}}{y^{1/3}}.
\]
Next we have, differentiating implicitly again, that
\[
\frac{d}{dx} \left[ \frac{d}{dx} \right] = \frac{d}{dx} \left[ -\frac{y^{1/3}}{x^{1/3}} \right]
\]
\[
\frac{d^2y}{dx^2} = \left[ [(1/3)y^{-2/3} \left( \frac{dy}{dx} \right)] \left( x^{1/3} \right) - (y^{1/3}) \left( 1/3 \right) x^{-2/3} \right] \quad \text{or} \quad \left( x^{1/3} \right) ^2
\]

Exercise 3.7.22 (continued).

Exercise 3.7.44. Normals Parallel to a Line.
Find the normals to the curve \( xy + 2x - y = 0 \) that are parallel to the line \( 2x + y = 0 \).

**Solution (continued).** So we look for a point \((x_1, y_1)\) such that
\[
\ldots x_1 = -2y_1 - 3. \quad \text{Since } (x_1, y_1) \text{ lies on the curve } xy + 2x - y = 0, \text{ then we need } (-2y_1 - 3)y_1 + 2(-2y_1 - 3) - y_1 = 0 \quad \text{or} \quad -2y_1^2 - 3y_1 - 4y_1 - 6 - y_1 = 0 \quad \text{or} \quad -2y_1^2 - 8y_1 - 6 = 0 \quad \text{or} \quad y_1^2 + 4y_1 + 3 = 0 \quad \text{or} \quad (y_1 + 3)(y_1 + 1) = 0. \quad \text{So we need } y_1 = -3 \quad \text{or} \quad y_1 = -1 \quad \text{and then } (from \quad xy + 2x - y = 0) \text{ we have } x_1(-3) + 2x_1(-3) - y_1 = 0 \quad \text{or} \quad x_1(-1) + 2x_1(-1) = 0 \quad \text{and so } x_1 = -1, \quad \text{respectively. That is, the desired normal lines occur at the points } (3, -3) \text{ and } (-1, -1). \quad \text{Since the slope of the normal line is } m = -2 \text{ then by the point-slope formula the two normal lines are } y = -(3)(x - (-3)) \text{ or } y = -(1)(x - (-1)) \text{ or } \left[ y = -2x - 3 \right].
Exercise 3.7.48

Exercise 3.7.48. The Folium of Descartes. 
(a) Find the slope of the folium of Descartes \(x^3 + y^3 - 9xy = 0\) at the points \((4, 2)\) and \((2, 4)\). 
(b) At what point other than the origin does the folium have a horizontal tangent? 
(c) Find the coordinates of the point \(A\) in Figure 3.29 where the folium has a vertical tangent.

Solution. (a) Differentiating implicitly we have 
\[
\frac{d}{dx}[x^3 + y^3 - 9xy] = \frac{d}{dx}[0] \quad \text{or} \quad 3x^2 + 3y^2 \frac{dy}{dx} - 9 \left[(1)(y) + (x) \frac{dy}{dx}\right] = 0 \quad \text{or} \quad (3y^2 - 9x) \frac{dy}{dx} = 9y - 3x^2
\]
\[
\ldots
\]

(b) At what point other than the origin does the folium have a horizontal tangent?

Solution (continued). (b) If the folium has a horizontal tangent at 
\[(x, y) = (x_1, y_1)\] 
then 
\[
\left.\frac{dy}{dx}\right|_{(x, y) = (x_1, y_1)} = \frac{9y_1 - 3x_1^2}{3y_1^2 - 9x_1} = 0. \quad \text{This implies} \quad 9y_1 - 3x_1^2 = 0, \quad \text{or} \quad y_1 = x_1^2/3. \quad \text{Since} \quad (x_1, y_1) \text{ also lies on the folium} \quad x^3 + y^3 - 9xy = 0, \quad \text{then we must have} \quad x_1^3 + (x_1^2/3)^3 - 9x_1(x_1^2/3) = 0 \quad \text{or} \quad x_1^2 + x_1^6/27 - 3x_1^3 = 0 \quad \text{or} \quad x_1^6/27 - 2x_1^3 = 0 \quad \text{or} \quad x_1^3(x_1^2/27 - 2) = 0. \quad \text{So we need either} \quad x_1 = 0 \quad \text{or} \quad x_1 = 3^{1/2}. \quad \text{When} \quad x_1 = 0 \quad \text{then} \quad y_1 = 0 \quad \text{and this is the horizontal tangent at the origin.} \quad \text{When} \quad x_1 = 3^{1/2} \quad \text{we have} \quad y_1 = (3^{1/2})^2/3 = 9(2^{1/2})^3/3 = 3(2^{3/2}) = 3^{3/2}. \quad \text{So the other horizontal tangent occurs at} \quad (x_1, y_1) = (3^{1/2}, 3^{3/2}) \quad \square\]

(c) Find the coordinates of the point \(A\) in Figure 3.29 where the folium has a vertical tangent.

Solution (continued). (c) Since \[
\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x},
\] then we look for a vertical tangent where the denominator is 0 and the numerator is not (see Dr. Bob’s Infinite Limits Theorem in 2.6. Limits Involving Infinity: Asymptotes of Graphs and the definition of vertical tangent line in 3.1. Tangent Lines and the Derivative at a Point). So if the folium has a vertical tangent at \((x_1, y_1)\) then \(3y_1^2 - 9x_1 = 0, \quad \text{or} \quad x_1 = y_1^2/3. \quad \text{Since} \quad (x_1, y_1) \text{ also lies on the folium} \quad x^3 + y^3 - 9xy = 0, \quad \text{then we must have} \quad (y_1^2/3)^3 + y_1^3 - 9(y_1^2/3)y_1 = 0 \quad \text{or} \quad y_1^6/27 + y_1^3 - 3y_1^3 = 0 \quad \text{or} \quad y_1^2/27 - 2y_1^3 = 0 \quad \text{or} \quad y_1^3(y_1^2/27 - 2) = 0. \quad \text{So we need either} \quad y_1 = 0 \quad \text{or} \quad y_1 = 3^{1/2}. \quad \text{When} \quad y_1 = 0 \quad \text{then} \quad x_1 = 0 \quad \text{so that} \quad (x_1, y_1) = (0, 0) \quad \text{and we see that} \quad \text{the folium has both a horizontal tangent and a vertical tangent at the origin.} \quad \text{When} \quad y_1 = 3^{1/2} \quad \text{we have} \quad x_1 = (3^{1/2})^2/3 = 3^{1/2}/3 = 3^{1/2}. \quad \square
Exercise 3.7.48 (continued 4)

(c) Find the coordinates of the point A in Figure 3.29 where the folium has a vertical tangent.

Solution (continued). We see from the graph that the folium has a vertical tangent at point A (where the x coordinate is near 5). We know that if the folium has a vertical tangent then it occurs at \((x_1, y_1) = (3\sqrt{4}, 3\sqrt{2})\) (or at the origin \((x_1, y_1) = (0, 0)\)).

so it must be that \(\text{point A is } (3\sqrt{4}, 3\sqrt{2})\).

We have \(3\sqrt{4} \approx 4.76\), consistent with the graph. Also notice that the horizontal tangent is at \((3\sqrt{2}, 3\sqrt{4})\) by (b), reflecting the symmetry of the folium with respect to the line \(y = x\). \(\square\)

Exercise 3.7.50

Exercise 3.7.50. Power Rule for Rational Exponents.

Let \(p\) and \(q\) be integers with \(q > 0\). If \(y = x^{p/q}\), differentiate the equivalent equation \(y^q = x^p\) implicitly and show that, for \(y \neq 0\),

\[
\frac{d}{dx}[x^{p/q}] = \frac{p}{q} x^{(p/q)-1}.
\]

Solution. Now \(y = x^{p/q}\) if and only if \(y^q = (x^{p/q})^q = x^p\), so differentiating implicitly we have \(\frac{d}{dx}[y^q] = \frac{d}{dx}[x^p]\) or (since \(q > 0\) and \(y \neq 0\))

\[
q y^{q-1} \frac{dy}{dx} = px^{p-1} \text{ or } \frac{dy}{dx} = \frac{px^{p-1}}{q y^{q-1}} \text{ or, since } y = x^{p/q},
\]

\[
\frac{dy}{dx} = \frac{px^{p-1}}{q(x^{p/q})^{q-1}} \text{ or } \frac{dy}{dx} = \frac{px^{p-1}}{q x^{p-1} - (p-1)/q} \text{ or }
\]

\[
\frac{dy}{dx} = \frac{p}{q} x^{(p/q)-1}, \text{ as claimed. } \square
\]