Exercise 3.8.8. Let \( f(x) = x^2 - 4x - 5, x > 2 \). Find the value of \( df^{-1}/dx \) at the point \( x = 0 = f(5) \).

Solution. By Theorem 3.3, The Derivative Rule for Inverses, we have

\[
\frac{df^{-1}}{dx}
\bigg|_{x=b} = \frac{1}{\frac{df}{dx}
\bigg|_{x=f^{-1}(b)}}.
\]

Here, \( b = 0, f^{-1}(b) = f^{-1}(0) = 5 \), and \( \frac{df}{dx} = 2x - 4 \). So we have

\[
\frac{df^{-1}}{dx}
\bigg|_{x=0} = \frac{1}{2x-4}
\bigg|_{x=f^{-1}(b)=f^{-1}(0)=5} = \frac{1}{2(5)-4} = \frac{1}{6}.
\]

\[ \square \]

Theorem 3.3. The Derivative Rule for Inverses

If \( f \) has an interval \( I \) as its domain and \( f'(x) \) exists and is never zero on \( I \), then \( f^{-1} \) is differentiable at every point in its domain. The value of \( (f^{-1})' \) at a point \( b \) in the domain of \( f^{-1} \) is the reciprocal of the value of \( f' \) at the point \( a = f^{-1}(b) \):

\[
\frac{df^{-1}}{dx}
\bigg|_{x=b} = \frac{1}{\frac{df}{dx}
\bigg|_{x=f^{-1}(b)}}.
\]

Proof. By definition of inverse function, \( f^{-1}(f(x)) = x \) for all \( x \in I \). Differentiating this equation, we have by the Chain Rule (Theorem 3.2):

\[
\frac{d}{dx} [f^{-1}(f(x))] = \frac{d}{dx} [x] \text{ or } f^{-1}'(f(x))^2 = 1 \text{ or } f^{-1}'(f(x)) = \frac{1}{f'(x)}.
\]

Plugging in \( x = f^{-1}(b) \) we get \( f^{-1}'(f^{-1}(b)) = \frac{1}{f'(f^{-1}(b))} \), as claimed.

\[ \square \]

Theorem 3.8.A

3.8.A. For \( x > 0 \) we have

\[
\frac{d}{dx} [\ln x] = \frac{1}{x}.
\]

If \( u = u(x) \) is a differentiable function of \( x \), then for all \( x \) such that \( u(x) > 0 \) we have

\[
\frac{d}{dx} [\ln u] = \frac{d}{dx} [\ln u(x)] = \frac{1}{u(x)} \frac{du}{dx}.
\]

Proof. We know that \( f(x) = e^x \) is differentiable for all \( x \), so we can apply Theorem 3.3 to find the derivative of \( f^{-1}(x) = \ln x \):

\[
\frac{d}{dx} [\ln x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x},
\]

as claimed.
Theorem 3.8.A (continued)

Theorem 3.8.A. For $x > 0$ we have

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$ 

If $u = u(x)$ is a differentiable function of $x$, then for all $x$ such that $u(x) > 0$ we have

$$\frac{d}{dx} \ln u = \frac{d}{dx} \ln u(x) = \frac{1}{u} \left[ \frac{du}{dx} \right] = \frac{1}{u(x)} \left[ u'(x) \right].$$

Proof (continued). By the Chain Rule (Theorem 3.2),

$$\frac{d}{dx} \ln u(x) = \frac{d}{du} \ln u \left[ \frac{du}{dx} \right] = \frac{1}{u} \left[ \frac{du}{dx} \right],$$

as claimed.

Exercise 3.8.16

Exercise 3.8.16. Find $\frac{dy}{dx}$ when $y = \ln(\sin x)$.

Solution. By Theorem 3.8.A,

$$\frac{dy}{dx} = \frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \cos x = \frac{\cos x}{\sin x} = \cot x.$$

Exercise 3.8.30

Exercise 3.8.30. Find $\frac{dy}{dx}$ when $y = \ln(\ln(x))$.

Solution. We have three “levels” of functions, a natural logarithm inside a natural logarithm inside another natural logarithm. So we will have to use the Chain Rule (Theorem 3.2) twice. We have

$$\frac{dy}{dx} = \frac{d}{dx} \ln(\ln x) = \frac{1}{\ln x} \left[ \frac{1}{x} \right] = \frac{1}{x \ln(x) \ln(x)}.$$
Exercise 3.8.52

Exercise 3.8.52. Find $y'$ by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\frac{(x + 1)^{10}}{(2x + 1)^5}}$.

**Solution.** First, we have

$$
\ln y = \ln \left( \sqrt{\frac{(x + 1)^{10}}{(2x + 1)^5}} \right) = \ln \left( \frac{(x + 1)^{10}}{(2x + 1)^5} \right)^{1/2} = \frac{1}{2} \ln \left( \frac{(x + 1)^{10}}{(2x + 1)^5} \right)
$$

$$
= \frac{1}{2} \ln(x + 1)^{10} - \frac{1}{2} \ln(2x + 1)^5 = \frac{1}{2} \left( 10 \ln(x + 1) - 5 \ln(2x + 1) \right)
$$

$$
= 5 \ln(x + 1) - \frac{5}{2} \ln(2x + 1).
$$

Now we differentiate implicitly:

$$
\frac{d}{dx} [\ln y] = \frac{d}{dx} \left[ 5 \ln(x + 1) - \frac{5}{2} \ln(2x + 1) \right]
$$

Exercise 3.8.52 (continued 2)

Exercise 3.8.52. Find $y'$ by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\frac{(x + 1)^{10}}{(2x + 1)^5}}$.

**Solution.** ...

$$
\frac{d}{dx} [\ln y] = \frac{1}{y} \left[ \frac{dy}{dx} \right] = \frac{5}{x + 1} - \frac{5}{2x + 1},
$$

and hence

$$
\frac{dy}{dx} = y \left( \frac{5}{x + 1} - \frac{5}{2x + 1} \right) = \sqrt{\frac{(x + 1)^{10}}{(2x + 1)^5}} \left( \frac{5}{x + 1} - \frac{5}{2x + 1} \right).
$$


Theorem 3.8.B

**Theorem 3.8.B.** If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$
\frac{d}{dx} [a^u] = (\ln a) a^u \left[ \frac{du}{dx} \right].
$$

**Proof.** First

$$
\frac{d}{dx} [a^x] = \frac{d}{dx} [e^{x \ln a}] = e^{x \ln a} \left[ \frac{d}{dx} [x \ln a] \right] = a^x \ln a = (\ln a) a^x.
$$

Then be the Chain Rule (Theorem 3.2),

$$
\frac{d}{dx} [a^u] = \frac{da^u}{du} \left[ \frac{du}{dx} \right] = (\ln a) a^u \left[ \frac{du}{dx} \right],
$$

as claimed.


Exercise 3.8.70

Exercise 3.8.70. Find \( dy/dx \) when \( y = 2^{(x^2)} \).

Solution. By Theorem 3.8.8 (with \( a = 2 \) and \( u(x) = x^2 \)), we have:

\[
\frac{d}{dx}[y] = \frac{dy}{dx} = \frac{d}{dx}[2^{(x^2)}] = (\ln 2)2^{(x^2)}[2x] = (2\ln 2)x2^{(x^2)}
\]

\( \square \)

Exercise 3.8.80

Exercise 3.8.80. Find \( dy/d\theta \) when \( y = \log_5\left(\frac{7x}{3x+2}\right)^{\ln 5} \).

Solution. We first apply some properties of logarithms:

\[
y = \log_5\left(\frac{7x}{3x+2}\right)^{\ln 5} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln 5}{2} \log_5\left(\frac{7x}{3x+2}\right)
\]

So by Theorem 3.8.C (with \( a = 5 \), \( u(x) = 7x \), and \( u_2(x) = 3x + 2 \)) we have

\[
\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\ln 5}{2} \left(\log_5(7x) - \log_5(3x + 2)\right)\right]
\]

\( = \frac{\ln 5}{2} \left(\frac{d}{dx}\log_5(7x) - \frac{d}{dx}\log_5(3x + 2)\right)\)

\( \square \)
Exercise 3.8.80. Find $dy/dx$ when $y = \log_5 \sqrt[\ln 5]{\frac{7x}{3x + 2}}$.

Solution. ...

\[
\frac{dy}{dx} = \frac{\ln 5}{2} \left( \frac{d}{dx} \left[ \log_5(7x) \right] - \frac{d}{dx} \left[ \log_5(3x + 2) \right] \right) = \frac{\ln 5}{2} \left( \frac{1}{\ln 5} \frac{1}{7x} - \frac{1}{\ln 5} \frac{1}{3x + 2} \right) = \frac{1}{2} \left( \frac{1}{x} - \frac{3}{3x + 2} \right).
\]

Exercise 3.8.90. Use logarithmic differentiation to find $dy/dx$: $y = x^{x+1}$.

Solution. Notice that $y$ has $x$ in both the base and the exponent, so that it is neither an exponential function nor a power of $x$. We must take a logarithm and use logarithmic differentiation. First, we have

\[
\ln y = \ln x^{x+1} = (x + 1) \ln x.
\]

Then

\[
\frac{d}{dx} \left[ \ln y \right] = \frac{d}{dx} \left[ (x + 1) \ln x \right]
\]

or

\[
\frac{1}{y} \frac{dy}{dx} = (x + 1) \left( \frac{1}{x} \right) \ln x + \frac{x}{x + 1}.
\]

so

\[
\frac{dy}{dx} = x^{x+1} \left( \frac{\ln x + \frac{x}{x + 1}}{x} \right).
\]
Example 3.8.72. Differentiate $y = t^{1-e}$.

Solution. This is an easy problem computationally, but we do it at this time because the exponent $1 - e$ is irrational. By Theorem 3.3.C/3.8.D, “General Power Rule for Derivatives,” we have

$$\frac{dy}{dt} = \frac{d}{dt}[t^{1-e}] = (1-e)t^{(1-e)-1} = (1-e)t^{-e}.$$ 

\[ \square \]

Theorem 3.4. The Number $e$ as a Limit

We can find $e$ as a limit:

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$ 

Proof. Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f'(1) = 1$. Now by the definition of derivative:

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{x \to 0} \frac{f(1+x) - f(1)}{x} = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x).$$

$$= \ln \left( \lim_{x \to 0} (1 + x)^{1/x} \right) \text{ since } \ln x \text{ is continuous.}$$

Exercise 3.8.102

Exercise 3.8.102. Show that $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Solution. As in the proof of Theorem 3.4, “The Number $e$ as a Limit,” we let $f(x) = \ln x$ (this is where we need $x > 0$) so that $f'(x) = 1/x$ and by the definition of derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}. $$

Now the exponential function is continuous at all real numbers, so

$$e^{1/x} = e^{\lim_{h \to 0} (\ln(x+h) - \ln x)/h} = \lim_{h \to 0} e^{\ln((x+h)/x) - \ln x/h} = \lim_{h \to 0} e^{(1/h)\ln((x+h)/x)}$$

$$= \lim_{h \to 0} e^{\ln((x+h)/x)/h} = \lim_{h \to 0} \left(\frac{x+h}{x}\right)^{1/h} = \lim_{h \to 0} \left(1 + \frac{h}{x}\right)^{1/h}. $$
Exercise 3.8.102. Show that \( \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x \) for any \( x > 0 \).

Solution (continued). \( e^{1/x} = \lim_{h \to 0} \left( 1 + \frac{h}{x} \right)^{1/h} \). In particular, we have \( e^{1/x} = \lim_{h \to 0^+} \left( 1 + \frac{h}{x} \right)^{1/h} \). Replacing \( h \) with \( 1/n \) and noting that \( h \to 0^+ \) if and only if \( n \to \infty \), we then have \( e^{1/x} = \lim_{n \to \infty} \left( 1 + \frac{1}{nx} \right)^n \). Now replacing \( x \) with \( 1/x \) we get \( e^x = \lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n \), as claimed. \( \square \)