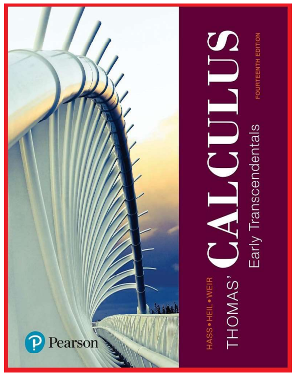
Calculus 1

Chapter 3. Derivatives

3.8. Derivatives of Inverse Functions and Logarithms—Examples and Proofs



Theorem 3.3

Theorem 3.3. The Derivative Rule for Inverses
If f has an interval I as its domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Proof. By definition of inverse function, $f^{-1}(f(x)) = x$ for all $x \in I$. Differentiating this equation, we have by the Chain Rule (Theorem 3.2):
$\frac{d}{dx}\left[f^{-1}(f(x))\right] = \frac{d}{dx}[x]$ or $f^{-1\prime}(f(x))[f'(x)] = 1$ or $f^{-1\prime}(f(x)) = \frac{1}{f'(x)}$.
Plugging in $x = f^{-1}(b)$ we get $f^{-1\prime}(f(f^{-1}(b))) = \frac{1}{f'(f^{-1}(b))}$, as claimed. □

Exercise 3.8.8

Exercise 3.8.8. Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find the value of df^{-1}/dx at the point $x = 0 = f(5)$.

Solution. By Theorem 3.3, The Derivative Rule for Inverses, we have

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Here, $b = 0$, $f^{-1}(b) = f^{-1}(0) = 5$, and $\frac{df}{dx} = 2x - 4$. So we have

$$\left.\frac{df^{-1}}{dx}\right|_{x=b=0} = \frac{1}{2x-4|_{x=f^{-1}(b)=f^{-1}(0)=5}} = \frac{1}{2(5)-4} = \boxed{\frac{1}{6}}.$$

□

Theorem 3.8.A

Theorem 3.8.A. For $x > 0$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\ln u(x)] = \frac{1}{u}\left[\frac{du}{dx}\right] = \frac{1}{u(x)}\left[u'(x)\right].$$

Proof. We know that $f(x) = e^x$ is differentiable for all x, so we can apply Theorem 3.3 to find the derivative of $f^{-1}(x) = \ln x$:

$$\frac{d}{dx}[\ln x] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x},$$

as claimed.

Theorem 3.8.A (continued)

Theorem 3.8.A. For $x > 0$ we have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}.$$

If $u = u(x)$ is a differentiable function of x, then for all x such that $u(x) > 0$ we have

$$\frac{d}{dx}[\ln u] = \frac{d}{dx}[\ln u(x)] = \frac{1}{u}\left[\frac{du}{dx}\right] = \frac{1}{u(x)}[u'(x)].$$

Proof (continued). By the Chain Rule (Theorem 3.2),

$$\frac{d}{dx}[\ln u(x)] = \frac{d}{du}[\ln u]\left[\frac{du}{dx}\right] = \frac{1}{u}\left[\frac{du}{dx}\right],$$

as claimed. □

Exercise 3.8.16

Exercise 3.8.16. Find dy/dx when $y = \ln(\sin x)$.

Solution. By Theorem 3.8.A,

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(\sin x)] = \frac{1}{\sin x}[\cos x] = \frac{\cos x}{\sin x} = \boxed{\cot x}.$$

□

Exercise 3.8.30

Exercise 3.8.30. Find dy/dx when $y = \ln(\ln(\ln x))$.

Solution. We have three "levels" of functions, a natural logarithm inside a natural logarithm inside another natural logarithm. So we will have to use the Chain Rule (Theorem 3.2) twice. We have

$$\frac{dy}{dx} = \frac{d}{dx}[\ln(\ln(\ln x))] = \frac{1}{\ln(\ln x)}\left[\frac{1}{\ln x}\left[\frac{1}{x}\right]\right] = \boxed{\frac{1}{x\ln(x)\ln(\ln(x))}}.$$

□

Exercise 3.8.38

Exercise 3.8.38. Find $dy/d\theta$ when $y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right)$.

Solution. First, we use properties of logarithms to modify the form of y:

$$y = \ln\left(\frac{\sqrt{\sin\theta\cos\theta}}{1+2\ln\theta}\right) = \ln\sqrt{\sin\theta\cos\theta} - \ln(1+2\ln\theta)$$

$$= \ln(\sin\theta\cos\theta)^{1/2} - \ln(1+2\ln\theta) = \frac{1}{2}\ln(\sin\theta\cos\theta) - \ln(1+2\ln\theta)$$

$$= \frac{1}{2}\ln(\sin\theta) + \frac{1}{2}\ln(\cos\theta) - \ln(1+2\ln\theta)$$

$$\frac{dy}{d\theta} = \frac{1}{2}\frac{1}{\sin\theta}[\cos\theta] + \frac{1}{2}\frac{1}{\cos\theta}[-\sin\theta] - \frac{1}{1+2\ln\theta}\left[0 + 2\frac{1}{\theta}\right]$$

$$= \boxed{\frac{1}{2}\cot\theta - \frac{1}{2}\tan\theta - \frac{2}{\theta(1+2\ln\theta)}}.$$ □

Exercise 3.8.52

Exercise 3.8.52. Find y' by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$.

Solution. First, we have

$$\ln y = \ln\left(\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\right) = \ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{(x+1)^{10}}{(2x+1)^5}\right)$$

$$= \frac{1}{2}\left(\ln(x+1)^{10} - \ln(2x+1)^5\right) = \frac{1}{2}\left(10\ln(x+1) - 5\ln(2x+1)\right)$$

$$= 5\ln(x+1) - \frac{5}{2}\ln(2x+1).$$

Now we differentiate implicitly:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[5\ln(x+1) - \frac{5}{2}\ln(2x+1)\right]$$

Exercise 3.8.52 (continued 1)

Exercise 3.8.52. Find y' by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$.

Solution. Now we differentiate implicitly:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[5\ln(x+1) - \frac{5}{2}\ln(2x+1)\right]$$

$$= 5\frac{d}{dx}[\ln(x+1)] - \frac{5}{2}\frac{d}{dx}[\ln(2x+1)] = 5\frac{1}{x+1}[1] - \frac{5}{2}\frac{1}{2x+1}[2]$$

$$= \frac{5}{x+1} - \frac{5}{2x+1}.$$

So

$$\frac{d}{dx}[\ln y] = \frac{1}{y}\left[\frac{dy}{dx}\right] = \frac{5}{x+1} - \frac{5}{2x+1},$$

Exercise 3.8.52 (continued 2)

Exercise 3.8.52. Find y' by first taking a natural logarithm and then differentiating implicitly: $y = \sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}$.

Solution. ...

$$\frac{d}{dx}[\ln y] = \frac{1}{y}\left[\frac{dy}{dx}\right] = \frac{5}{x+1} - \frac{5}{2x+1},$$

and hence

$$\frac{dy}{dx} = y\left(\frac{5}{x+1} - \frac{5}{2x+1}\right) = \boxed{\sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}\left(\frac{5}{x+1} - \frac{5}{2x+1}\right)}.$$

□

Theorem 3.8.B

Theorem 3.8.B. If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}[a^u] = (\ln a)a^u\left[\frac{du}{dx}\right].$$

Proof. First

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{x\ln a}\right] = e^{x\ln a}\left[\frac{d}{dx}[x\ln a]\right] = a^x\ln a = (\ln a)a^x.$$

Then be the Chain Rule (Theorem 3.2),

$$\frac{d}{dx}[a^u] = \frac{da^u}{du}\left[\frac{du}{dx}\right] = (\ln a)a^u\left[\frac{du}{dx}\right],$$

as claimed. □

Exercise 3.8.70

Exercise 3.8.70. Find dy/dx when $y = 2^{(x^2)}$.

Solution. By Theorem 3.8.B (with $a = 2$ and $u(x = x^2)$, we have:

$$\frac{d}{dx}[y] = \frac{dy}{dx} = \frac{d}{dx}[2^{(x^2)}] = (\ln 2)2^{(x^2)}[2x] = \boxed{(2\ln 2)x2^{(x^2)}}.$$

□

Theorem 3.8.C

Theorem 3.8.C. Differentiating a logarithm base a gives:

$$\frac{d}{dx}[\log_a u] = \frac{1}{\ln a}\frac{1}{u}\left[\frac{du}{dx}\right].$$

Proof. This follows easily:

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{\ln x}{\ln a}\right] = \frac{1}{\ln a}\frac{d}{dx}[\ln x] = \frac{1}{\ln a}\frac{1}{x}.$$

Then be the Chain Rule (Theorem 3.2),

$$\frac{d}{dx}[\log_a u] = \frac{d\log_a u}{du}\left[\frac{du}{dx}\right] = \frac{1}{\ln a}\frac{1}{u}\left[\frac{du}{dx}\right],$$

as claimed. □

Exercise 3.8.74

Exercise 3.8.74. Find $dy/d\theta$ when $y = \log_3(1 + \theta\ln 3)$.

Solution. By Theorem 3.3.C (with $a = 3$ and $u(\theta) = 1 + \theta\ln 3$) we have:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}[\log_3(1+\theta\ln 3)] = \frac{1}{\ln 3}\frac{1}{1+\theta\ln 3}[0 + \ln 3] = \boxed{\frac{1}{1+\theta\ln 3}}.$$

□

Exercise 3.8.80

Exercise 3.8.80. Find dy/dx when $y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$.

Solution. We first apply some properties of logarithms:

$$y = \log_5\sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}} = \log_5\left(\frac{7x}{3x+2}\right)^{(\ln 5)/2} = \frac{\ln 5}{2}\log_5\frac{7x}{3x+2}$$

$$= \frac{\ln 5}{2}\left(\log_5(7x) - \log_5(3x+2)\right).$$

So by Theorem 3.8.C (with $a = 5$, $u_1(x) = 7x$, and $u_2(x) = 3x + 2$) we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\ln 5}{2}\left(\log_5(7x) - \log_5(3x+2)\right)\right]$$

$$= \frac{\ln 5}{2}\left(\frac{d}{dx}[\log_5(7x)] - \frac{d}{dx}[\log_5(3x+2)]\right)$$

Exercise 3.8.80 (continued)

Exercise 3.8.80. Find dy/dx when $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$.

Solution. ...

$$\begin{aligned}\frac{dy}{dx} &= \frac{\ln 5}{2}\left(\frac{d}{dx}[\log_5(7x)] - \frac{d}{dx}[\log_5(3x+2)]\right)\\ &= \frac{\ln 5}{2}\left(\frac{1}{\ln 5}\frac{1}{7x}[7] - \frac{1}{\ln 5}\frac{1}{3x+2}[3]\right)\\ &= \boxed{\frac{1}{2}\left(\frac{1}{x} - \frac{3}{3x+2}\right)}.\end{aligned}$$

□

Exercise 3.8.90

Exercise 3.8.90. Use logarithmic differentiation to find dy/dx: $y = x^{x+1}$.

Solution. Notice that y has x in both the base *and* the exponent, so that it is neither an exponential function nor a power of x. We must take a logarithm and use logarithmic differentiation. First, we have $\ln y = \ln x^{x+1} = (x+1)\ln x$. Then $\frac{d}{dx}[\ln y] = \frac{d}{dx}[(x+1)\ln x]$ or $\frac{1}{y}\left[\frac{dy}{dx}\right] = [1](\ln x) + (x+1)\left[\frac{1}{x}\right]$ or $\frac{dy}{dx} = y\left(\ln x + \frac{x+1}{x}\right)$, so

$$\boxed{\frac{dy}{dx} = x^{x+1}\left(\ln x + \frac{x+1}{x}\right)}. \quad \square$$

Theorem 3.3.C/3.8.D

Theorem 3.3.C/3.8.D. General Power Rule for Derivatives.
For $x > 0$ and any real number n,

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

If $x < 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof. We have for $x > 0$ that

$$\begin{aligned}\frac{d}{dx}[x^n] &= \frac{d}{dx}\left[e^{n\ln x}\right]\\ &= e^{n\ln x}\frac{d}{dx}[n\ln x] \text{ by the Chain Rule}\\ &= x^n\frac{n}{x} = nx^{n-1},\end{aligned}$$

as claimed.

Theorem 3.3.C/3.8.D (continued)

Proof (continued). When $x < 0$, if $y = x^n$, y', and x^{n-1} all exist, then we have $\ln|y| = \ln|x^n| = \ln|x|^n = n\ln|x|$. Differentiating implicitly (this is where we must assume that y' exists) we have that $\frac{d}{dx}[\ln|y|] = \frac{d}{dx}[n\ln|x|]$, which implies (by Example 3.8.3(c)) $\frac{1}{y}\left[\frac{dy}{dx}\right] = n\frac{1}{x}$, or $\frac{dy}{dx} = ny\frac{1}{x} = nx^n\frac{1}{x} = nx^{n-1}$, as claimed.

This still leaves the case that for $x = 0$ and $n \geq 1$, the derivative is 0; this is to be shown in Exercise 3.8.103. □

Exercise 3.8.72

Example 3.8.72. Differentiate $y = t^{1-e}$.

Solution. This is an easy problem computationally, but we do it at this time because the exponent $1 - e$ is irrational. By Theorem 3.3.C/3.8.D, "General Power Rule for Derivatives," we have

$$\frac{dy}{dt} = \frac{d}{dt}[t^{1-e}] = (1-e)t^{(1-e)-1} = \boxed{(1-e)t^{-e}}.$$

□

Theorem 3.4

Theorem 3.4. The Number e as a Limit
We can find e as a limit:

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof. Let $f(x) = \ln x$. Then $f'(x) = 1/x$ and $f'(1) = 1$. Now by the definition of derivative:

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{\ln(1+x)-\ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x) \\ &= \lim_{x\to 0}\ln(1+x)^{1/x} \\ &= \ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) \text{ since } \ln x \text{ is continuous.} \end{aligned}$$

Theorem 3.4 (continued)

Theorem 3.4. The Number e as a Limit
We can find e as a limit:

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

Proof (continued). Therefore, since $f'(1) = 1$, we have

$$\ln\left(\lim_{x\to 0}(1+x)^{1/x}\right) = 1.$$

Since $\ln e = 1$ and $\ln x$ is one-to-one,

$$\lim_{x\to 0}(1+x)^{1/x} = e.$$

□

Exercise 3.8.102

Exercise 3.8.102. Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Solution. As in the proof of Theorem 3.4, "The Number e as a Limit," we let $f(x) = \ln x$ (this is where we need $x > 0$) so that $f'(x) = 1/x$ and by the definition of derivative,

$$\frac{1}{x} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}.$$

Now the exponential function is continuous at all real numbers, so

$$e^{1/x} = e^{\lim_{h\to 0}(\ln(x+h)-\ln x)/h} = \lim_{h\to 0} e^{(\ln(x+h)-\ln x)/h} = \lim_{h\to 0} e^{(1/h)\ln((x+h)/x)}$$

$$= \lim_{h\to 0} e^{\ln((x+h)/x)^{1/h}} = \lim_{h\to 0}\left(\frac{x+h}{x}\right)^{1/h} = \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}.$$

Exercise 3.8.102 (continued)

Exercise 3.8.102. Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x > 0$.

Solution (continued). $\ldots e^{1/x} = \lim_{h\to 0}\left(1+\frac{h}{x}\right)^{1/h}$. In particular, we have $e^{1/x} = \lim_{h\to 0^+}\left(1+\frac{h}{x}\right)^{1/h}$. Replacing h with $1/n$ and noting that $h \to 0^+$ if and only if $n \to \infty$, we then have $e^{1/x} = \lim_{n\to\infty}\left(1+\frac{1}{nx}\right)^n$. Now replacing x with $1/x$ we get $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, as claimed. $\square$