Chapter 3. Derivatives
3.9. Inverse Trigonometric Functions—Examples and Proofs
Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: (a) \( \sin^{-1}(1/2) \), (b) \( \sin^{-1}(-1/\sqrt{2}) \), (c) \( \arcsin(\sqrt{3}/2) \).

Solution. (a) With \( \theta = \sin^{-1}(1/2) \), we need \( \sin \theta = 1/2 \) and \( \theta \in [-\pi/2, \pi/2] \). So \( \theta \) is a “special angle” and from our knowledge of special angles, we have \( \theta = \pi/6 \). □

(b) With \( \theta = \sin^{-1}(-1/\sqrt{2}) \), we need \( \sin \theta = -1/\sqrt{2} = -\sqrt{2}/2 \) and \( \theta \in [-\pi/2, \pi/2] \). From our knowledge of special angles, we know that \( \sin \pi/4 = \sqrt{2}/2 \). So we seek an angle \( \theta \) with a reference angle of \( \pi/4 \) where \( \theta \in [-\pi/2, \pi/2] \) and \( \sin \theta < 0 \). We take \( \theta = -\pi/4 \): □

(c) With \( \theta = \arcsin(\sqrt{3}/2) \), we need \( \sin \theta = \sqrt{3}/2 \) and \( \theta \in [-\pi/2, \pi/2] \). From our knowledge of special angles, we have \( \theta = \pi/3 \). □
Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: (a) $\sin^{-1}(1/2)$, (b) $\sin^{-1}(-1/\sqrt{2})$, (c) $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin \theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So $\theta$ is a “special angle” and from our knowledge of special angles, we have $\theta = \pi/6$.

(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin \theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin \pi/4 = \sqrt{2}/2$. So we seek an angle $\theta$ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin \theta < 0$. We take $\theta = -\pi/4$.
Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: (a) $\sin^{-1}(1/2)$, (b) $\sin^{-1}(-1/\sqrt{2})$, (c) $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin \theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So $\theta$ is a “special angle” and from our knowledge of special angles, we have $\theta = \pi/6$. □

(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin \theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin \pi/4 = \sqrt{2}/2$. So we seek an angle $\theta$ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin \theta < 0$. We take $\theta = -\pi/4$. □
Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: (a) \( \sin^{-1}(1/2) \), (b) \( \sin^{-1}(-1/\sqrt{2}) \), (c) \( \arcsin(\sqrt{3}/2) \).

Solution. (a) With \( \theta = \sin^{-1}(1/2) \), we need \( \sin \theta = 1/2 \) and \( \theta \in [-\pi/2, \pi/2] \). So \( \theta \) is a “special angle” and from our knowledge of special angles, we have \( \theta = \pi/6 \). □

(b) With \( \theta = \sin^{-1}(-1/\sqrt{2}) \), we need \( \sin \theta = -1/\sqrt{2} = -\sqrt{2}/2 \) and \( \theta \in [-\pi/2, \pi/2] \). From our knowledge of special angles, we know that \( \sin \pi/4 = \sqrt{2}/2 \). So we seek an angle \( \theta \) with a reference angle of \( \pi/4 \) where \( \theta \in [-\pi/2, \pi/2] \) and \( \sin \theta < 0 \). We take \( \theta = -\pi/4 \): □
Exercise 3.9.4 (continued)

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: (a) \( \sin^{-1}(1/2) \), (b) \( \sin^{-1}(-1/\sqrt{2}) \), (c) \( \arcsin(\sqrt{3}/2) \).

Solution. (c) With \( \theta = \arcsin(\sqrt{3}/2) \), we need \( \sin \theta = \sqrt{3}/2 \) and \( \theta \in [-\pi/2, \pi/2] \). So \( \theta \) is a “special angle” and from our knowledge of special angles, we have \( \theta = \pi/3 \). □
Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: 
(a) \(\sin^{-1}(1/2)\), (b) \(\sin^{-1}(-1/\sqrt{2})\), (c) \(\arcsin(\sqrt{3}/2)\).

Solution. (c) With \(\theta = \arcsin(\sqrt{3}/2)\), we need \(\sin \theta = \sqrt{3}/2\) and \(\theta \in [-\pi/2, \pi/2]\). So \(\theta\) is a “special angle” and from our knowledge of special angles, we have \(\theta = \pi/3\). □
Exercise 3.9.14

Exercise 3.9.14. Find the limit: \( \lim_{x \to -1^+} \cos^{-1}(x) \).

Solution. First, notice that \(-1\) is a left endpoint of the domain of \( \cos^{-1} x \).
Based on the graph of \( y = \cos^{-1} x \), we see (by Dr. Bob’s Anthropomorphic Definition of Limit, a one-sided version) that as \( x \to -1 \) from the right (i.e., from the positive side) that the graph “tries to contain the point” \((-1, \pi)\). So
\[
\lim_{x \to -1^+} \cos^{-1}(x) = \pi.
\]
Exercise 3.9.14. Find the limit: \( \lim_{x \to -1^+} \cos^{-1}(x) \).

**Solution.** First, notice that \(-1\) is a left endpoint of the domain of \( \cos^{-1} x \).

Based on the graph of \( y = \cos^{-1} x \), we see (by Dr. Bob’s Anthropomorphic Definition of Limit, a one-sided version) that as \( x \to -1 \) from the right (i.e., from the positive side) that the graph “tries to contain the point” \((-1, \pi)\). So \( \lim_{x \to -1^+} \cos^{-1}(x) = \pi \). \( \square \)
Exercise 3.9.14. Find the limit: \( \lim_{x \to -1^+} \cos^{-1}(x) \).

Solution. First, notice that \(-1\) is a left endpoint of the domain of \( \cos^{-1} x \). Based on the graph of \( y = \cos^{-1} x \), we see (by Dr. Bob’s Anthropomorphic Definition of Limit, a one-sided version) that as \( x \to -1 \) from the right (i.e., from the positive side) that the graph “tries to contain the point” \((-1, \pi)\). So \( \lim_{x \to -1^+} \cos^{-1}(x) = \pi \). \( \square \)
Theorem 3.9.A

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$
\frac{d}{dx} \left[ \sin^{-1} u \right] = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}
$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$$
\frac{d}{dx} [\sin y] = \frac{d}{dx} [x] \text{ or } \cos y \frac{dy}{dx} = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}.
$$
Theorem 3.9.A

**Theorem 3.9.A.** We differentiate $\sin^{-1}$ as follows:

$$
\frac{d}{dx} \left[ \sin^{-1} u \right] = \frac{1}{\sqrt{1 - u^2}} \left[ \frac{du}{dx} \right]
$$

where $|u| < 1$.

**Proof.** We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$$
\frac{d}{dx} [\sin y] = \frac{d}{dx} [x] \text{ or } \cos y \left[ \frac{dy}{dx} \right] = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}.
$$

Since we have restricted $y$ to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1 - (\sin y)^2} = \sqrt{1 - x^2}$. Making this substitution we get

$$
\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}.
$$

The full theorem then follows from the Chain Rule.
**Theorem 3.9.A**

**Theorem 3.9.A.** We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx} [\sin^{-1} u] = \frac{1}{\sqrt{1 - u^2}} \left[ \frac{du}{dx} \right]$$

where $|u| < 1$.

**Proof.** We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$$\frac{d}{dx} [\sin y] = \frac{d}{dx} [x] \text{ or } \cos y \left[ \frac{dy}{dx} \right] = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}.$$  Since we have restricted $y$ to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1 - (\sin y)^2} = \sqrt{1 - x^2}$. Making this substitution we get

$$\frac{d}{dx} [\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}.$$  The full theorem then follows from the Chain Rule.
Exercise 3.9.24. For $dy/dt$ when $y = \sin^{-1}(1 - t)$.

Solution. By Theorem 3.9.A (with $u(t) = 1 - t$ and $du/dt = -1$), we have

$$
\frac{dy}{dt} = \frac{d}{dt}[\sin^{-1}(1 - t)] = \frac{1}{\sqrt{1 - (1 - t)^2}}[-1] = \frac{-1}{\sqrt{2t - t^2}}.
$$
Exercise 3.9.24. For \( \frac{dy}{dt} \) when \( y = \sin^{-1}(1 - t) \).

Solution. By Theorem 3.9.A (with \( u(t) = 1 - t \) and \( du/dt = -1 \)), we have

\[
\frac{dy}{dt} = \frac{d}{dt}[\sin^{-1}(1 - t)] = \frac{1}{\sqrt{1 - (1 - t)^2}}[-1] = \frac{-1}{\sqrt{2t - t^2}}.
\]
Theorem 3.9.B. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx} \left[ \tan^{-1} u \right] = \frac{1}{1 + u^2} \left[ \frac{du}{dx} \right].$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation

$$\frac{d}{dx} [\tan y] = \frac{d}{dx} [x] \text{ or } \sec^2 y \left[ \frac{dy}{dx} \right] = 1 \text{ or }$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + (\tan y)^2} = \frac{1}{1 + x^2}.$$ The full theorem then follows from the Chain Rule. □
Theorem 3.9.B. We differentiate \( \tan^{-1} \) as follows:

\[
\frac{d}{dx} \left[ \tan^{-1} u \right] = \frac{1}{1 + u^2} \left[ \frac{du}{dx} \right].
\]

Proof. We know that if \( y = \tan^{-1} x \) then (for appropriate domain and range values) \( \tan y = x \) and so by implicit differentiation

\[
\frac{d}{dx} \left[ \tan y \right] = \frac{d}{dx} [x] \text{ or } \sec^2 y \left[ \frac{dy}{dx} \right] = 1 \text{ or }
\]

\[
\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + (\tan y)^2} = \frac{1}{1 + x^2}. \text{ The full theorem then follows from the Chain Rule.}\]
Exercise 3.9.34. Find \( \frac{dy}{dx} \) when \( y = \tan^{-1}(\ln x) \).

Solution. By Theorem 3.9.B (with \( u(x) = \ln x \) and \( \frac{du}{dx} = \frac{1}{x} \)), we have

\[
\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\ln x)] = \frac{1}{1 + (\ln x)^2} \left[ \frac{1}{x} \right] = \frac{1}{x(1 + (\ln x)^2)}.
\]
Exercise 3.9.34. Find $dy/dx$ when $y = \tan^{-1}(\ln x)$.

Solution. By Theorem 3.9.B (with $u(x) = \ln x$ and $du/dx = 1/x$), we have

$$
\frac{dy}{dx} = \frac{d}{dx} \left[ \tan^{-1}(\ln x) \right] = \frac{1}{1 + (\ln x)^2} \left[ \frac{1}{x} \right] = \frac{1}{x(1 + (\ln x)^2)}.
$$
Theorem 3.9.C

Theorem 3.9.C. We differentiate $\sec^{-1}$ as follows:

$$
\frac{d}{dx} \left[ \sec^{-1} u \right] = \frac{1}{|u|\sqrt{u^2 - 1}} \left[ \frac{du}{dx} \right]
$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation

$$
\frac{d}{dx} [\sec y] = \frac{d}{dx} [x] \text{ or } \sec y \tan y \left[ \frac{dy}{dx} \right] = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\sec y \tan y}.
$$

We now need to express this last expression in terms of $x$. First, $\sec y = x$ and $\tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$. Therefore we have

$$
\frac{d}{dx} \left[ \sec^{-1} x \right] = \pm \frac{1}{x\sqrt{x^2 - 1}}.
$$
Theorem 3.9.C

**Theorem 3.9.C.** We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx} \left[ \sec^{-1} u \right] = \frac{1}{|u| \sqrt{u^2 - 1}} \left[ \frac{du}{dx} \right]$$

where $|u| > 1$.

**Proof.** We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation

$$\frac{d}{dx} \left[ \sec y \right] = \frac{d}{dx} [x] \text{ or } \sec y \tan y \left[ \frac{dy}{dx} \right] = 1 \text{ or } \frac{dy}{dx} = \frac{1}{\sec y \tan y}.$$  

We now need to express this last expression in terms of $x$. First, $\sec y = x$ and $\tan y = \pm \sqrt{\sec^2 y - 1} = \pm \sqrt{x^2 - 1}$. Therefore we have

$$\frac{d}{dx} \left[ \sec^{-1} x \right] = \pm \frac{1}{x \sqrt{x^2 - 1}}.$$
Theorem 3.9.C (continued)

Proof (continued). . .

\[
\frac{d}{dx} [\sec^{-1} x] = \pm \frac{1}{x\sqrt{x^2 - 1}}.
\]

Notice from the graph of \( y = \sec^{-1} x \) above, that the slope of this function is positive wherever it is defined. So

\[
\frac{d}{dx} [\sec^{-1} x] = \begin{cases} 
+ \frac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\
- \frac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1.
\end{cases}
\]

Notice that if \( x > 1 \) then \( x = |x| \) and if \( x < -1 \) then \( -x = |x| \). Therefore

\[
\frac{d}{dx} [\sec^{-1} x] = \frac{1}{|x|\sqrt{x^2 - 1}}.
\]

The full theorem then follows from the Chain Rule.
Theorem 3.9.C (continued)

Proof (continued). . . .

\[
\frac{d}{dx} \left[ \sec^{-1} x \right] = \pm \frac{1}{x \sqrt{x^2 - 1}}.
\]

Notice from the graph of \( y = \sec^{-1} x \) above, that the slope of this function is positive wherever it is defined. So

\[
\frac{d}{dx} \left[ \sec^{-1} x \right] = \begin{cases} 
+ \frac{1}{x \sqrt{x^2 - 1}} & \text{if } x > 1 \\
- \frac{1}{x \sqrt{x^2 - 1}} & \text{if } x < -1.
\end{cases}
\]

Notice that if \( x > 1 \) then \( x = |x| \) and if \( x < -1 \) then \( -x = |x| \). Therefore

\[
\frac{d}{dx} \left[ \sec^{-1} x \right] = \frac{1}{|x| \sqrt{x^2 - 1}}.
\]

The full theorem then follows from the Chain Rule.
Exercise 3.9.40. Find $dy/dx$ when $y = \cot^{-1}(1/x) - \tan^{-1} x$.

Solution. By Table 3.1(3 and 4) (with $u(x) = 1/x = x^{-1}$ and $du/dx = -x^{-2} = -1/x^2$), we have

$$
\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(1/x) - \tan^{-1} x] = \frac{d}{dx}[\cot^{-1}(1/x)] - \frac{d}{dx}[\tan^{-1} x]
$$

$$
= \frac{-1}{1 + (1/x)^2} \left[ -\frac{1}{x^2} \right] - \frac{1}{1 + x^2}
$$

$$
= \frac{1}{x^2(1 + 1/x^2)} - \frac{1}{1 + x^2} = \frac{1}{x^2 + 1} - \frac{1}{1 + x^2} = 0.
$$
Exercise 3.9.40. Find $dy/dx$ when $y = \cot^{-1}(1/x) - \tan^{-1} x$.

Solution. By Table 3.1(3 and 4) (with $u(x) = 1/x = x^{-1}$ and $du/dx = -x^{-2} = -1/x^2$), we have

$$
\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(1/x) - \tan^{-1} x] = \frac{d}{dx}[\cot^{-1}(1/x)] - \frac{d}{dx}[\tan^{-1} x]
$$

$$
= -\frac{1}{1 + (1/x)^2} \left[ -\frac{1}{x^2} \right] - \frac{1}{1 + x^2}
$$

$$
= -\frac{x^2}{x^2(1 + 1/x^2)} - \frac{1}{1 + x^2} = \frac{1}{x^2 + 1} - \frac{1}{1 + x^2} = 0.
$$

□
Exercise 3.9.44. Find $dy/dx$ at point $P(0, 1/2)$ when $\sin^{-1}(x + y) + \cos^{-1}(x - y) = 5\pi/6$.

Solution. Differentiating implicitly we have by Table 3.1(1 and 2) that

$$\frac{d}{dx} [\sin^{-1}(x + y) + \cos^{-1}(x - y)] = \frac{d}{dx} \left[ \frac{5\pi}{6} \right] \text{ or }$$

$$\frac{d}{dx} [\sin^{-1}(x + y)] + \frac{d}{dx} [\cos^{-1}(x - y)] = \frac{d}{dx} \left[ \frac{5\pi}{6} \right] \text{ or }$$

$$\frac{1}{\sqrt{1 - (x + y)^2}} \left( 1 + \frac{dy}{dx} \right) + \frac{-1}{\sqrt{1 - (x - y)^2}} \left( 1 - \frac{dy}{dx} \right) = 0 \text{ or }$$

$$\left( \frac{1}{\sqrt{1 - (x + y)^2}} + \frac{1}{\sqrt{1 - (x - y)^2}} \right) \frac{dy}{dx} = \frac{-1}{\sqrt{1 - (x + y)^2}} + \frac{1}{\sqrt{1 - (x - y)^2}} \text{ or }$$

(getting a common denominator)
Exercise 3.9.44

Exercise 3.9.44. Find \( \frac{dy}{dx} \) at point \( P(0, 1/2) \) when \( \sin^{-1}(x + y) + \cos^{-1}(x - y) = \frac{5\pi}{6} \).

Solution. Differentiating implicitly we have by Table 3.1(1 and 2) that

\[
\frac{d}{dx}[\sin^{-1}(x + y) + \cos^{-1}(x - y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \quad \text{or}
\]

\[
\frac{d}{dx}[\sin^{-1}(x + y)] + \frac{d}{dx}[\cos^{-1}(x - y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \quad \text{or}
\]

\[
\frac{1}{\sqrt{1 - (x + y)^2}} \left[1 + \frac{dy}{dx}\right] + \frac{-1}{\sqrt{1 - (x - y)^2}} \left[1 - \frac{dy}{dx}\right] = 0 \quad \text{or}
\]

\[
\left(\frac{1}{\sqrt{1 - (x + y)^2}} + \frac{1}{\sqrt{1 - (x - y)^2}}\right) \frac{dy}{dx} = \frac{-1}{\sqrt{1 - (x + y)^2}} + \frac{1}{\sqrt{1 - (x - y)^2}}
\]

(getting a common denominator)
Exercise 3.9.44. Find \( \frac{dy}{dx} \) at point \( P(0, 1/2) \) when \( \sin^{-1}(x + y) + \cos^{-1}(x - y) = 5\pi/6 \).

Solution (continued). \( \ldots \) 

\[
\left( \frac{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}{\sqrt{1 - (x + y)^2} \sqrt{1 - (x - y)^2}} \right) \frac{dy}{dx} = -\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}
\]

or

\[
\left( \frac{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}} \right) \frac{dy}{dx} = -\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}
\]

With \((x, y) = (0, 1/2)\) we have \( \sqrt{1 - (x \pm y)^2} = \sqrt{3/4} = \sqrt{3}/2 \) and at \( P(0, 1/2) \) we then have \( \frac{dy}{dx}\big|_{(x,y)=(0,1/2)} = 0 \). \( \square \)
Exercise 3.9.44 (continued)

**Exercise 3.9.44.** Find $dy/dx$ at point $P(0,1/2)$ when $\sin^{-1}(x + y) + \cos^{-1}(x - y) = 5\pi/6$.

**Solution (continued).** ... \[
\left( \frac{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}{\sqrt{1 - (x + y)^2} \sqrt{1 - (x - y)^2}} \right) \frac{dy}{dx} = \\
\frac{-\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}{\sqrt{1 - (x + y)^2} \sqrt{1 - (x - y)^2}}
\]

or

\[
\left( \sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2} \right) \frac{dy}{dx} = -\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2} \text{ or}
\]

\[
\frac{dy}{dx} = -\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2} \frac{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}{\sqrt{1 - (x - y)^2} + \sqrt{1 - (x + y)^2}}.
\]

With $(x, y) = (0, 1/2)$ we have $\sqrt{1 - (x \pm y)^2} = \sqrt{3/4} = \sqrt{3}/2$ and at $P(0, 1/2)$ we then have $dy/dx|_{(x,y)=(0,1/2)} = 0$. □
Exercise 3.9.60

**Exercise 3.9.60.** What is special about the functions \( f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} \) and \( g(x) = \tan^{-1}(1/x) \)?

**Solution.** Notice that

\[
\frac{df}{dx} = \frac{d}{dx} \left[ \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} \right] = \frac{1}{\sqrt{1 - (1/\sqrt{x^2 + 1})^2}} \frac{d}{dx} \left[ (x^2 + 1)^{-1/2} \right]
\]

\[
= \frac{1}{\sqrt{1 - (1/\sqrt{x^2 + 1})^2}} \left[ -\frac{1}{2} (x^2 + 1)^{-3/2} \frac{d}{dx} [2x] \right]
\]

\[
= \frac{1}{\sqrt{1 - 1/(x^2 + 1)}} (-x(x^2 + 1)^{-3/2}) = \frac{1}{\sqrt{(x^2 + 1) - 1}/(x^2 + 1)} \frac{-x}{(x^2 + 1)^{3/2}}
\]

\[
= \frac{\sqrt{x^2 + 1}}{\sqrt{x^2}} \frac{-x}{(x^2 + 1)\sqrt{x^2 + 1}} = \frac{-x}{|x|(x^2 + 1)}
\]
Exercise 3.9.60

Exercise 3.9.60. What is special about the functions

\[ f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} \quad \text{and} \quad g(x) = \tan^{-1}(1/x)? \]

Solution. Notice that

\[
\frac{df}{dx} = \frac{d}{dx} \left[ \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} \right] = \frac{1}{\sqrt{1 - (1/\sqrt{x^2 + 1})^2}} \frac{d}{dx} \left[ (x^2 + 1)^{-1/2} \right]
\]

\[
= \frac{1}{\sqrt{1 - (1/\sqrt{x^2 + 1})^2}} \left[ -\frac{1}{2} (x^2 + 1)^{-3/2} \frac{d}{dx} [2x] \right]
\]

\[
= \frac{1}{\sqrt{1 - 1/(x^2 + 1)}} (-x(x^2 + 1)^{-3/2}) = \frac{1}{\sqrt{(x^2 + 1) - 1/(x^2 + 1)}} (x^2 + 1)^{3/2}
\]

\[
= \frac{\sqrt{x^2 + 1}}{\sqrt{x^2}} \frac{-x}{(x^2 + 1)\sqrt{x^2 + 1}} = \frac{-x}{|x|(x^2 + 1)}
\]
Solution. Notice that

\[
\frac{dg}{dx} = \frac{d}{dx} \left[ \tan^{-1} \frac{1}{x} \right] = \frac{1}{1 + (1/x)^2} \frac{d}{dx} \left[ \frac{1}{x} \right] = \frac{1}{1 + (1/x)^2} \left[ -\frac{1}{x^2} \right]
\]

\[
= -\frac{1}{(1 + (1/x)^2)x^2} = -\frac{1}{x^2 + 1}.
\]

So for \( x > 0 \), \( f'(x) = g'(x) \). We will see in Corollary 4.2 (see Section 4.2. The Mean Value Theorem) that this implies \( f(x) - g(x) \) is constant. We can evaluate \( f \) and \( g \) at some \( x > 0 \) to see what this constant is. With \( x = 1 \) we have

\[
f(1) = \sin^{-1} \frac{1}{\sqrt{(1)^2 + 1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4 \text{ and}
\]

\[
g(1) = \tan^{-1}(1/(1)) = \tan^{-1}(1) = \pi/4, \text{ so that the constant is 0 and so we must have}
\]

\[
f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} = \tan^{-1}(1/x) = g(x) \text{ for } x > 0.
\]
Solution. Notice that

\[
\frac{dg}{dx} = \frac{d}{dx} \left[ \tan^{-1} \frac{1}{x} \right] = \frac{1}{1 + (1/x)^2} \frac{d}{dx} \left[ \frac{1}{x} \right] = \frac{1}{1 + (1/x)^2} \left\{ -1 \right\}
\]

\[
= \frac{-1}{(1 + (1/x)^2)x^2} = -\frac{1}{x^2 + 1}.
\]

So for \( x > 0 \), \( f'(x) = g'(x) \). We will see in Corollary 4.2 (see Section 4.2. The Mean Value Theorem) that this implies \( f(x) - g(x) \) is constant. We can evaluate \( f \) and \( g \) at some \( x > 0 \) to see what this constant is. With \( x = 1 \) we have

\[
f(1) = \sin^{-1} \frac{1}{\sqrt{(1)^2 + 1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4 \text{ and } \\
g(1) = \tan^{-1}(1/(1)) = \tan^{-1}(1) = \pi/4,
\]

so that the constant is 0 and so we must have

\[
f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} = \tan^{-1}(1/x) = g(x) \text{ for } x > 0.
\]
Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution (continued). For $x < 0$, $f'(x) = -g'(x)$ or $f'(x) + g'(x) = 0$. Again, by Corollary 4.2 (see Section 4.2. The Mean Value Theorem) this implies $f(x) + g(x)$ is constant. We can evaluate $f$ and $g$ at some $x < 0$ to see what this constant is. With $x = -1$ we have

$f(-1) = \sin^{-1} \frac{1}{\sqrt{(-1)^2 + 1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and

$g(-1) = \tan^{-1}(1/(-1)) = \tan^{-1}(-1) = -\pi/4$, so that

$f(x) + g(x) = \pi/4 + (-\pi/4) = 0$ for $x < 0$, or

$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} = -\tan^{-1}(1/x) = -g(x)$ for $x < 0$. □
Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution (continued). For $x < 0$, $f'(x) = -g'(x)$ or $f'(x) + g'(x) = 0$. Again, by Corollary 4.2 (see Section 4.2. The Mean Value Theorem) this implies $f(x) + g(x)$ is constant. We can evaluate $f$ and $g$ at some $x < 0$ to see what this constant is. With $x = -1$ we have $f(-1) = \sin^{-1} \frac{1}{\sqrt{(-1)^2 + 1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and $g(-1) = \tan^{-1}(1/(-1)) = \tan^{-1}(-1) = -\pi/4$, so that $f(x) + g(x) = \pi/4 + (-\pi/4) = 0$ for $x < 0$, or

$$f(x) = \sin^{-1} \frac{1}{\sqrt{x^2 + 1}} = -\tan^{-1}(1/x) = -g(x) \text{ for } x < 0. \quad \square$$