Calculus 1

Chapter 3. Derivatives

3.9. Inverse Trigonometric Functions—Examples and Proofs

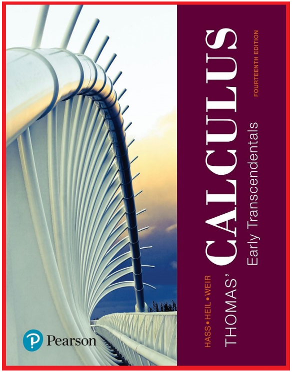


Table of contents

1. Exercise 3.9.4
2. Exercise 3.9.14
3. Theorem 3.9.A
4. Exercise 3.9.24
5. Theorem 3.9.B
6. Exercise 3.9.34
7. Theorem 3.9.C
8. Exercise 3.9.40
9. Exercise 3.9.44
10. Exercise 3.9.60

Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin\theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/6}$. □

Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin\theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/6}$. □

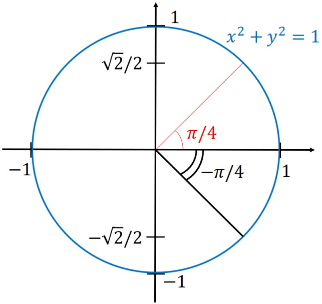
(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin\theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin\pi/4 = \sqrt{2}/2$. So we seek an angle θ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin\theta < 0$. We take $\boxed{\theta = -\pi/4}$.

Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin\theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/6}$. □

(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin\theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin\pi/4 = \sqrt{2}/2$. So we seek an angle θ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin\theta < 0$. We take $\boxed{\theta = -\pi/4}$.



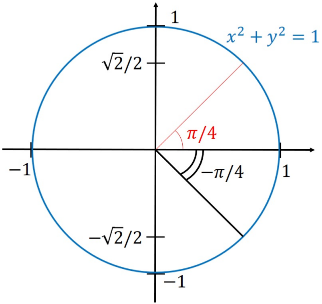
□

Exercise 3.9.4

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (a) With $\theta = \sin^{-1}(1/2)$, we need $\sin\theta = 1/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/6}$. □

(b) With $\theta = \sin^{-1}(-1/\sqrt{2})$, we need $\sin\theta = -1/\sqrt{2} = -\sqrt{2}/2$ and $\theta \in [-\pi/2, \pi/2]$. From our knowledge of special angles, we know that $\sin\pi/4 = \sqrt{2}/2$. So we seek an angle θ with a reference angle of $\pi/4$ where $\theta \in [-\pi/2, \pi/2]$ and $\sin\theta < 0$. We take $\boxed{\theta = -\pi/4}$.



□

Exercise 3.9.4 (continued)

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (c) With $\theta = \arcsin(\sqrt{3}/2)$, we need $\sin\theta = \sqrt{3}/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/3}$. □

Exercise 3.9.4 (continued)

Exercise 3.9.4. Use reference angles in an appropriate quadrant to find the angles: **(a)** $\sin^{-1}(1/2)$, **(b)** $\sin^{-1}(-1/\sqrt{2})$, **(c)** $\arcsin(\sqrt{3}/2)$.

Solution. (c) With $\theta = \arcsin(\sqrt{3}/2)$, we need $\sin\theta = \sqrt{3}/2$ and $\theta \in [-\pi/2, \pi/2]$. So θ is a "special angle" and from our knowledge of special angles, we have $\boxed{\theta = \pi/3}$. □

Exercise 3.9.14

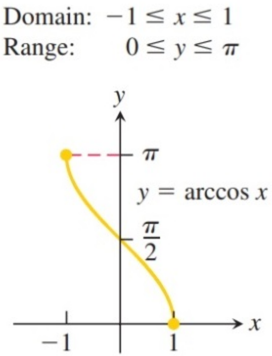
Exercise 3.9.14. Find the limit: $\lim_{x \to -1^+} \cos^{-1}(x)$.

Solution. First, notice that -1 is a left endpoint of the domain of $\cos^{-1} x$. Based on the graph of $y = \cos^{-1} x$, we see (by Dr. Bob's Anthropomorphic Definition of Limit, a one-sided version) that as $x \to -1$ from the right (i.e., from the positive side) that the graph "tries to contain the point" $(-1, \pi)$. So $\lim_{x \to -1^+} \cos^{-1}(x) = \boxed{\pi}$. □

Exercise 3.9.14

Exercise 3.9.14. Find the limit: $\lim_{x \to -1^+} \cos^{-1}(x)$.

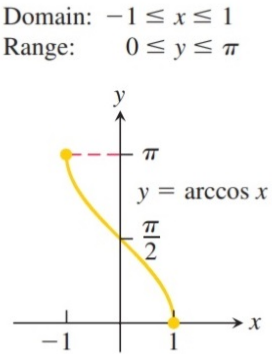
Solution. First, notice that -1 is a left endpoint of the domain of $\cos^{-1} x$. Based on the graph of $y = \cos^{-1} x$, we see (by Dr. Bob's Anthropomorphic Definition of Limit, a one-sided version) that as $x \to -1$ from the right (i.e., from the positive side) that the graph "tries to contain the point" $(-1, \pi)$. So $\lim_{x \to -1^+} \cos^{-1}(x) = \boxed{\pi}$. □



Exercise 3.9.14

Exercise 3.9.14. Find the limit: $\lim_{x \to -1^+} \cos^{-1}(x)$.

Solution. First, notice that -1 is a left endpoint of the domain of $\cos^{-1} x$. Based on the graph of $y = \cos^{-1} x$, we see (by Dr. Bob's Anthropomorphic Definition of Limit, a one-sided version) that as $x \to -1$ from the right (i.e., from the positive side) that the graph "tries to contain the point" $(-1, \pi)$. So $\lim_{x \to -1^+} \cos^{-1}(x) = \boxed{\pi}$. □



Theorem 3.9.A

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation

$\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$ or $\cos y \overset{\curvearrowright}{\left[\frac{dy}{dx}\right]} = 1$ or $\frac{dy}{dx} = \frac{1}{\cos y}$.

Theorem 3.9.A

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \left[\frac{du}{dx}\right]$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$ or $\cos y \left[\frac{dy}{dx}\right] = 1$ or $\frac{dy}{dx} = \frac{1}{\cos y}$. Since we have restricted y to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1-(\sin y)^2} = \sqrt{1-x^2}$. Making this substitution we get $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$. The full theorem then follows from the Chain Rule. □

Theorem 3.9.A

Theorem 3.9.A. We differentiate $\sin^{-1}$ as follows:

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| < 1$.

Proof. We know that if $y = \sin^{-1} x$ then (for appropriate domain and range values) $\sin y = x$ and so by implicit differentiation $\frac{d}{dx}[\sin y] = \frac{d}{dx}[x]$ or $\cos y \overset{\curvearrowright}{\left[\frac{dy}{dx}\right]} = 1$ or $\frac{dy}{dx} = \frac{1}{\cos y}$. Since we have restricted y to the interval $[-\pi/2, \pi/2]$, we know that $\cos y \geq 0$ and so $\cos y = +\sqrt{1-(\sin y)^2} = \sqrt{1-x^2}$. Making this substitution we get $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$. The full theorem then follows from the Chain Rule. □

Exercise 3.9.24

Exercise 3.9.24. For dy/dt when $y = \sin^{-1}(1-t)$.

Solution. By Theorem 3.9.A (with $u(t) = 1 - t$ and $du/dt = -1$), we have

$$\frac{dy}{dt} = \frac{d}{dt}[\sin^{-1}(1-t)] = \frac{1}{\sqrt{1-(1-t)^2}}[-1] = \boxed{\frac{-1}{\sqrt{2t-t^2}}}.$$

□

Exercise 3.9.24

Exercise 3.9.24. For dy/dt when $y = \sin^{-1}(1-t)$.

Solution. By Theorem 3.9.A (with $u(t) = 1 - t$ and $du/dt = -1$), we have

$$\frac{dy}{dt} = \frac{d}{dt}[\sin^{-1}(1-t)] = \frac{1}{\sqrt{1-(1-t)^2}}\overset{\curvearrowright}{[-1]} = \boxed{\frac{-1}{\sqrt{2t-t^2}}}.$$

□

Theorem 3.9.B

Theorem 3.9.B. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\left[\frac{du}{dx}\right].$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation

$\frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$ or $\sec^2 y \left[\frac{dy}{dx}\right] = 1$ or

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+(\tan y)^2} = \frac{1}{1+x^2}$. The full theorem then follows from the Chain Rule. □

Theorem 3.9.B

Theorem 3.9.B. We differentiate $\tan^{-1}$ as follows:

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{1+u^2}\overset{\curvearrowright}{} \left[\frac{du}{dx}\right].$$

Proof. We know that if $y = \tan^{-1} x$ then (for appropriate domain and range values) $\tan y = x$ and so by implicit differentiation

$\dfrac{d}{dx}[\tan y] = \dfrac{d}{dx}[x]$ or $\sec^2 y \overset{\curvearrowright}{} \left[\dfrac{dy}{dx}\right] = 1$ or

$\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+(\tan y)^2} = \dfrac{1}{1+x^2}$. The full theorem then follows from the Chain Rule. □

Exercise 3.9.34

Exercise 3.9.34. Find dy/dx when $y = \tan^{-1}(\ln x)$.

Solution. By Theorem 3.9.B (with $u(x) = \ln x$ and $du/dx = 1/x$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\ln x)] = \frac{1}{1+(\ln x)^2}\left[\frac{1}{x}\right] = \boxed{\frac{1}{x(1+(\ln x)^2}}.$$

□

Exercise 3.9.34

Exercise 3.9.34. Find dy/dx when $y = \tan^{-1}(\ln x)$.

Solution. By Theorem 3.9.B (with $u(x) = \ln x$ and $du/dx = 1/x$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\ln x)] = \frac{1}{1+(\ln x)^2}\overset{\curvearrowright}{\left[\frac{1}{x}\right]} = \boxed{\frac{1}{x(1+(\ln x)^2}}.$$

□

Theorem 3.9.C

Theorem 3.9.C. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation $\frac{d}{dx}[\sec y] = \frac{d}{dx}[x]$ or $\sec y \tan y \overset{\curvearrowright}{\left[\frac{dy}{dx}\right]} = 1$ or $\frac{dy}{dx} = \frac{1}{\sec y \tan y}$. We now need to express this last expression in terms of x. First, $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. Therefore we have

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Theorem 3.9.C

Theorem 3.9.C. We differentiate $\sec^{-1}$ as follows:

$$\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}}\overset{\curvearrowright}{\left[\frac{du}{dx}\right]}$$

where $|u| > 1$.

Proof. We know that if $y = \sec^{-1} x$ then (for appropriate domain and range values) $\sec y = x$ and so by implicit differentiation

$\displaystyle \frac{d}{dx}[\sec y] = \frac{d}{dx}[x]$ or $\displaystyle \sec y \tan y \overset{\curvearrowright}{\left[\frac{dy}{dx}\right]} = 1$ or $\displaystyle \frac{dy}{dx} = \frac{1}{\sec y \tan y}$. We now need to express this last expression in terms of x. First, $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$. Therefore we have

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Theorem 3.9.C (continued)

Proof (continued). ...

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Notice from the graph of $y = \sec^{-1} x$ above, that the slope of this function is positive wherever it is defined. So

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \begin{cases} +\frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

Notice that if $x > 1$ then $x = |x|$ and if $x < -1$ then $-x = |x|$. Therefore

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}.$$

The full theorem then follows from the Chain Rule. □

Theorem 3.9.C (continued)

Proof (continued). ...

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \pm\frac{1}{x\sqrt{x^2-1}}.$$

Notice from the graph of $y = \sec^{-1} x$ above, that the slope of this function is positive wherever it is defined. So

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \begin{cases} +\frac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\frac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

Notice that if $x > 1$ then $x = |x|$ and if $x < -1$ then $-x = |x|$. Therefore

$$\frac{d}{dx}\left[\sec^{-1} x\right] = \frac{1}{|x|\sqrt{x^2-1}}.$$

The full theorem then follows from the Chain Rule. □

Exercise 3.9.40

Exercise 3.9.40. Find dy/dx when $y = \cot^{-1}(1/x) - \tan^{-1} x$.

Solution. By Table 3.1(3 and 4) (with $u(x) = 1/x = x^{-1}$ and $du/dx = -x^{-2} = -1/x^2$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(1/x) - \tan^{-1} x] = \frac{d}{dx}[\cot^{-1}(1/x)] - \frac{d}{dx}[\tan^{-1} x]$$

$$= \frac{-1}{1+(1/x)^2}\left[\frac{-1}{x^2}\right] - \frac{1}{1+x^2}$$

$$= \frac{1}{x^2(1+1/x^2)} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = \boxed{0}.$$

□

Exercise 3.9.40

Exercise 3.9.40. Find dy/dx when $y = \cot^{-1}(1/x) - \tan^{-1} x$.

Solution. By Table 3.1(3 and 4) (with $u(x) = 1/x = x^{-1}$ and $du/dx = -x^{-2} = -1/x^2$), we have

$$\frac{dy}{dx} = \frac{d}{dx}[\cot^{-1}(1/x) - \tan^{-1} x] = \frac{d}{dx}[\cot^{-1}(1/x)] - \frac{d}{dx}[\tan^{-1} x]$$

$$= \frac{-1}{1+(1/x)^2} \overset{\curvearrowright}{} \left[\frac{-1}{x^2}\right] - \frac{1}{1+x^2}$$

$$= \frac{1}{x^2(1+1/x^2)} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = \boxed{0}.$$

□

Exercise 3.9.44

Exercise 3.9.44. Find dy/dx at point $P(0, 1/2)$ when $\sin^{-1}(x+y) + \cos^{-1}(x-y) = 5\pi/6$.

Solution. Differentiating implicitly we have by Table 3.1(1 and 2) that

$$\frac{d}{dx}[\sin^{-1}(x+y) + \cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \text{ or}$$

$$\frac{d}{dx}[\sin^{-1}(x+y)] + \frac{d}{dx}[\cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \text{ or}$$

$$\frac{1}{\sqrt{1-(x+y)^2}} \overset{\curvearrowright}{\left[1+\frac{dy}{dx}\right]} + \frac{-1}{\sqrt{1-(x-y)^2}} \overset{\curvearrowright}{\left[1-\frac{dy}{dx}\right]} = 0 \text{ or}$$

$$\left(\frac{1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\right)\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}} \text{ or}$$

(getting a common denominator)

Exercise 3.9.44

Exercise 3.9.44. Find dy/dx at point $P(0, 1/2)$ when $\sin^{-1}(x+y) + \cos^{-1}(x-y) = 5\pi/6$.

Solution. Differentiating implicitly we have by Table 3.1(1 and 2) that

$$\frac{d}{dx}[\sin^{-1}(x+y) + \cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \text{ or}$$

$$\frac{d}{dx}[\sin^{-1}(x+y)] + \frac{d}{dx}[\cos^{-1}(x-y)] = \frac{d}{dx}\left[\frac{5\pi}{6}\right] \text{ or}$$

$$\frac{1}{\sqrt{1-(x+y)^2}} \overset{\curvearrowright}{\left[1 + \frac{dy}{dx}\right]} + \frac{-1}{\sqrt{1-(x-y)^2}} \overset{\curvearrowright}{\left[1 - \frac{dy}{dx}\right]} = 0 \text{ or}$$

$$\left(\frac{1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}}\right)\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x+y)^2}} + \frac{1}{\sqrt{1-(x-y)^2}} \text{ or}$$

(getting a common denominator)

Exercise 3.9.44 (continued)

Exercise 3.9.44. Find dy/dx at point $P(0,1/2)$ when $\sin^{-1}(x+y)+\cos^{-1}(x-y)=5\pi/6$.

Solution (continued). ... $\left(\frac{\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}}\right)\frac{dy}{dx}=$ $\frac{-\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}}$ or $\left(\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}\right)\frac{dy}{dx}=-\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}$ or $\frac{dy}{dx}=\frac{-\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}}{\sqrt{1-(x-y)^2}+\sqrt{1-(x+y)^2}}$. With $(x,y)=(0,1/2)$ we have $\sqrt{1-(x\pm y)^2}=\sqrt{3/4}=\sqrt{3}/2$ and at $P(0,1/2)$ we then have $\boxed{dy/dx|_{(x,y)=(0,1/2)}=0}$. □

Exercise 3.9.44 (continued)

Exercise 3.9.44. Find dy/dx at point $P(0, 1/2)$ when $\sin^{-1}(x+y) + \cos^{-1}(x-y) = 5\pi/6$.

Solution (continued). $\ldots \left(\dfrac{\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}} \right) \dfrac{dy}{dx} = \dfrac{-\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^2}\sqrt{1-(x-y)^2}}$ or $\left(\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2} \right) \dfrac{dy}{dx} = -\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}$ or $\dfrac{dy}{dx} = \dfrac{-\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}{\sqrt{1-(x-y)^2} + \sqrt{1-(x+y)^2}}$. With $(x, y) = (0, 1/2)$ we have $\sqrt{1-(x \pm y)^2} = \sqrt{3/4} = \sqrt{3}/2$ and at $P(0, 1/2)$ we then have $\boxed{dy/dx|_{(x,y)=(0,1/2)} = 0}$. $\square$

Exercise 3.9.60

Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1}\dfrac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution. Notice that

$$\frac{df}{dx} = \frac{d}{dx}\left[\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right] = \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}}\overset{\curvearrowright}{\frac{d}{dx}}\left[(x^2+1)^{-1/2}\right]$$

$$= \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}}\overset{\curvearrowright}{\left[\frac{-1}{2}(x^2+1)^{-3/2}\overset{\curvearrowright}{[2x]}\right]}$$

$$= \frac{1}{\sqrt{1-1/(x^2+1)}}(-x(x^2+1)^{-3/2}) = \frac{1}{\sqrt{((x^2+1)-1)/(x^2+1)}}\frac{-x}{(x^2+1)^{3/2}}$$

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2}}\frac{-x}{(x^2+1)\sqrt{x^2+1}} = \frac{-x}{|x|(x^2+1)}$$

Exercise 3.9.60

Exercise 3.9.60. What is special about the functions $f(x) = \sin^{-1} \dfrac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution. Notice that

$$\frac{df}{dx} = \frac{d}{dx}\left[\sin^{-1}\frac{1}{\sqrt{x^2+1}}\right] = \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}}\overset{\curvearrowright}{\frac{d}{dx}}\left[(x^2+1)^{-1/2}\right]$$

$$= \frac{1}{\sqrt{1-(1/\sqrt{x^2+1})^2}}\overset{\curvearrowright}{}\left[\frac{-1}{2}(x^2+1)^{-3/2}\overset{\curvearrowright}{}[2x]\right]$$

$$= \frac{1}{\sqrt{1-1/(x^2+1)}}(-x(x^2+1)^{-3/2}) = \frac{1}{\sqrt{((x^2+1)-1)/(x^2+1)}}\frac{-x}{(x^2+1)^{3/2}}$$

$$= \frac{\sqrt{x^2+1}}{\sqrt{x^2}}\frac{-x}{(x^2+1)\sqrt{x^2+1}} = \frac{-x}{|x|(x^2+1)}$$

Exercise 3.9.60 (continued 1)

Solution. Notice that

$$\frac{dg}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\curvearrowright}{\frac{d}{dx}}\left[\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\curvearrowright}{\left[\frac{-1}{x^2}\right]}$$

$$= \frac{-1}{(1+(1/x)^2)x^2} = \frac{-1}{x^2+1}.$$

So for $x > 0$, $f'(x) = g'(x)$. We will see in Corollary 4.2 (see Section 4.2. The Mean Value Theorem) that this implies $f(x) - g(x)$ is constant. We can evaluate f and g at some $x > 0$ to see what this constant is. With $x = 1$ we have

$f(1) = \sin^{-1}\frac{1}{\sqrt{(1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and

$g(1) = \tan^{-1}(1/(1)) = \tan^{-1}(1) = \pi/4$, so that the constant is 0 and so

we must have $\boxed{f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}(1/x) = g(x) \text{ for } x > 0.}$

Exercise 3.9.60 (continued 1)

Solution. Notice that

$$\frac{dg}{dx} = \frac{d}{dx}\left[\tan^{-1}\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\curvearrowright}{\frac{d}{dx}}\left[\frac{1}{x}\right] = \frac{1}{1+(1/x)^2}\overset{\curvearrowright}{\left[\frac{-1}{x^2}\right]}$$

$$= \frac{-1}{(1+(1/x)^2)x^2} = \frac{-1}{x^2+1}.$$

So for $x > 0$, $f'(x) = g'(x)$. We will see in Corollary 4.2 (see Section 4.2. The Mean Value Theorem) that this implies $f(x) - g(x)$ is constant. We can evaluate f and g at some $x > 0$ to see what this constant is. With $x = 1$ we have
$f(1) = \sin^{-1}\frac{1}{\sqrt{(1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and
$g(1) = \tan^{-1}(1/(1)) = \tan^{-1}(1) = \pi/4$, so that the constant is 0 and so we must have $\boxed{f(x) = \sin^{-1}\frac{1}{\sqrt{x^2+1}} = \tan^{-1}(1/x) = g(x) \text{ for } x > 0.}$

Exercise 3.9.60 (continued 2)

Exercise 3.9.60. What is special about the functions
$f(x) = \sin^{-1} \dfrac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution (continued). For $x < 0$, $f'(x) = -g'(x)$ or $f'(x) + g'(x) = 0$. Again, by Corollary 4.2 (see Section 4.2. The Mean Value Theorem) this implies $f(x) + g(x)$ is constant. We can evaluate f and g at some $x < 0$ to see what this constant is. With $x = -1$ we have
$f(-1) = \sin^{-1} \frac{1}{\sqrt{(-1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and
$g(-1) = \tan^{-1}(1/(-1)) = \tan^{-1}(-1) = -\pi/4$, so that
$f(x) + g(x) = \pi/4 + (-\pi/4) = 0$ for $x < 0$, or

$$\boxed{f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} = -\tan^{-1}(1/x) = -g(x) \text{ for } x < 0.}$$ □

Exercise 3.9.60 (continued 2)

Exercise 3.9.60. What is special about the functions
$f(x) = \sin^{-1} \dfrac{1}{\sqrt{x^2+1}}$ and $g(x) = \tan^{-1}(1/x)$?

Solution (continued). For $x < 0$, $f'(x) = -g'(x)$ or $f'(x) + g'(x) = 0$. Again, by Corollary 4.2 (see Section 4.2. The Mean Value Theorem) this implies $f(x) + g(x)$ is constant. We can evaluate f and g at some $x < 0$ to see what this constant is. With $x = -1$ we have
$f(-1) = \sin^{-1} \frac{1}{\sqrt{(-1)^2+1}} = \sin^{-1}(1/\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$ and
$g(-1) = \tan^{-1}(1/(-1)) = \tan^{-1}(-1) = -\pi/4$, so that
$f(x) + g(x) = \pi/4 + (-\pi/4) = 0$ for $x < 0$, or

$$\boxed{f(x) = \sin^{-1} \frac{1}{\sqrt{x^2+1}} = -\tan^{-1}(1/x) = -g(x) \text{ for } x < 0.}$$ □