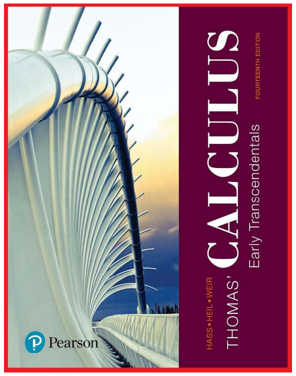
Calculus 1

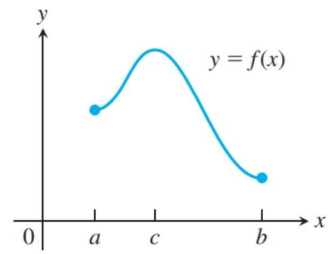
Chapter 4. Applications of Derivatives

4.1. Extreme Values of Functions on Closed Intervals—Examples and Proofs



Exercise 4.1.2

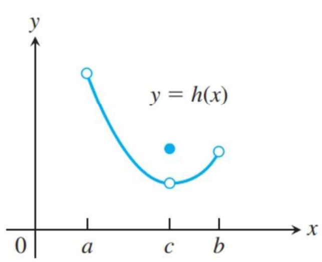
Exercise 4.1.2. Determine from the graph whether f has any absolute extreme values on $[a, b]$:



Solution. First, f is continuous on $[a, b]$ so by Theorem 4.1, The Extreme-Value Theorem for Continuous Functions, it has both an absolute maximum and absolute minimum. From the graph, we see that f has an absolute maximum of $f(c)$ and an absolute minimum of $f(b)$. □

Exercise 4.1.4

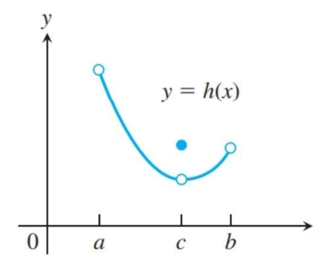
Exercise 4.1.4. Determine from the graph whether h has any absolute extreme values on $[a, b]$:



Solution. First, h is not defined on $[a, b]$, since h is not defined at $x = a$ nor at $x = b$. In addition, h is not defined at $x = c$. So Theorem 4.1 does not apply. In fact, h has neither an absolute maximum nor an absolute minimum.

Exercise 4.1.4 (continued)

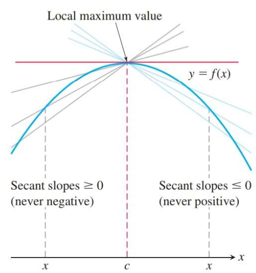
Solution (continued). We see that $\lim_{x\to a^+} h(x)$ exists and is strictly greater than any value of $h(x)$ for $x \in (a, b)$, and $\lim_{x\to c} h(x)$ exists and is strictly less than any value of $h(x)$ for $x \in (a, b)$. So these values are upper and lower bounds on the values of h, but neither value is attained by h on (a, b). In fact, values of h can be made arbitrarily close to both of these values (by making x sufficiently close to a and greater than a for the upper bound $\lim_{x\to a^+} h(x)$, and by making x sufficiently close to c for the lower bound $\lim_{x\to c} h(x)$). This is related to the idea that there is not a least positive real number (nor a greatest negative real number); remember that 0 is neither positive nor negative. . . because it is too busy being 0! □



Theorem 4.2

Theorem 4.2. Local Extreme Values.
If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then $f'(c) = 0$.

Proof. Suppose that f has a local maximum value at $x = c$, so that $f(x) - f(c) \le 0$ for all values of x in some open interval containing c. Since c is an interior point of the domain of f, then $f'(c)$ is (by the alternative definition of the derivative; see Exercise 3.2.24) $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. Considering one-sided limits and the fact that $f(c)$ is a local maximum of f, we have $f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$ since $f(x) - f(c) \le 0$ and for $x \to c^+$ we have $x - c > 0$, and $f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0$ since $f(x) - f(c) \le 0$ and for $x \to c^-$ we have $x - c < 0$.



Theorem 4.2 (continued)

Theorem 4.2. Local Extreme Values.
If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then $f'(c) = 0$.

Proof (continued). Since the two-sided limit exists, then the one-sided limits must both exist and be the same by Theorem 2.6. ("Relation Between One-Sided and Two-Sided Limits"), so we must have $f'(c) = 0$.

The argument when f has a local minimum value at $x = c$ (we then have $f(x) - f(c) \ge 0$ for all values of x in some open interval containing c and the inequalities in the one-sided limits are reversed) is similar. □

Exercise 4.1.24

Exercise 4.1.24. Find the absolute maximum and minimum values of $f(x) = 4 - x^3$ on the interval $[-2, 1]$. Then graph $y = f(x)$ and identify the points on the graph where the absolute extrema occur.

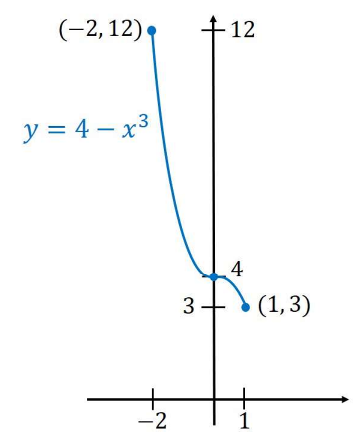
Solution. We follow the three steps just introduced. With $f(x) = 4 - x^3$, we have $f'(x) = -3x^2$ and for Step 1 we set $f'(x) = -3x^2 = 0$ and see that $x = 0$ is the only critical point. For Step 2, we consider the values of f at the critical point $x = 0$ and the endpoints $a = -2$ and $b = 1$:

x	-2	0	1
f(x)	$4 - (-2)^3 = 12$	$4 - (0)^3 = 4$	$4 - (1)^3 = 3$

By Step 3, the absolute maximum is 12 and occurs at $x = -2$, and the absolute minimum is 3 and occurs at $x = 1$.

Exercise 4.1.24 (continued)

Solution (continued). The graph is:



□

Exercise 4.1.44

Exercise 4.1.44. Find the absolute maximum and minimum values of $h(\theta) = 3\theta^{2/3}$ on the interval $[-27, 8]$.

Solution. We follow the three steps. With $h(\theta) = 3\theta^{2/3}$, we have $h'(\theta) = 3(2/3)\theta^{-1/3} = \dfrac{2}{\sqrt[3]{\theta}}$ and for Step 1 we see that h' is never 0, but h' is undefined at $\theta = 0$. So $\theta = 0$ is the only critical point. For Step 2, we consider the values of h at the critical point $\theta = 0$ and the endpoints $a = -27$ and $b = 8$:

θ	-27	0	8
$\mathbf{h}(\theta)$	$3(-27)^{2/3} = 27$	$3(0)^{2/3} = 0$	$3(8)^{2/3} = 12$

By Step 3, the absolute maximum is 27 and occurs at $\theta = -27$, and the absolute minimum is 0 and occurs at $\theta = 0$. □

Exercise 4.1.60

Exercise 4.1.60. Find the critical points and domain endpoints for $y = f(x) = x^2\sqrt{3-x}$. Then find the value of the function at each of these points and identify extreme values (absolute and local).

Solution. First, notice that the domain of f is $(-\infty, 3]$ (that is, $x \le 3$ where $3 - x \ge 0$), so 3 is an endpoint of the domain. Also, f is nonnegative. Since the domain is not an interval of the form $[a, b]$, we cannot precisely follow the three steps. But we still need the critical points of $f(x) = x^2(3-x)^{1/2}$ and so consider

$$f'(x) = [2x]((3-x)^{1/2}) + (x^2)[(1/2)(3-x)^{-1/2}[-1]] = 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}} = 2x\sqrt{3-x}\left(\frac{2\sqrt{3-x}}{2\sqrt{3-x}}\right) - \frac{x^2}{2\sqrt{3-x}} = \frac{4x(3-x)-x^2}{2\sqrt{3-x}} = \frac{12x-5x^2}{2\sqrt{3-x}} = \frac{x(12-5x)}{2\sqrt{3-x}}.$$

The critical points are $x = 0$ (because $f'(0) = 0$), $x = 12/5$ (because $f'(12/5) = 0$), and $x = 3$ (because $x = 3$ is in the domain of f but f' is not defined at $x = 3$).

Exercise 4.1.60 (continued 1)

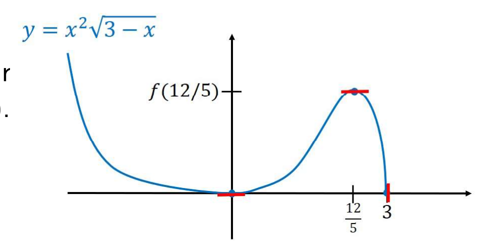
Solution (continued). We consider the values of f at the critical points and endpoint:

$\mathbf{x}$	0	$12/5$	3
$\mathbf{f(x)}$	$(0)^2\sqrt{3-(0)} = 0$	$(12/5)^2\sqrt{3-12/5} = (144/25)\sqrt{3/5}$	$(3)^2\sqrt{3-(3)} = 0$

Since $f(x) \ge 0$ for all x in its domain, then f must have an absolute minimum at $x = 0$ and $x = 3$ of 0. Next, we claim that f has a local maximum at $x = 12/5$. This is because $12/5$ is between 0 and 3, and $f(12/5) > f(0) = f(3)$; for if f had a larger value than $f(12/5)$ for some $0 < x < 3$, then (since f is differentiable for $0 < x < 3$) by Theorem 4.2, Local Extreme Values, f would have another critical point between 0 and 3 where the derivative is 0, but there is no such point. So $f(12/5)$ must be the largest value of f on the open interval $(0, 3)$ and hence f has a local maximum at $x = 12/5$ of $(144/25)\sqrt{3/5}$.

Exercise 4.1.60 (continued 2)

Solution (continued). As shown above, $f'(x) = \dfrac{x(12-5x)}{2\sqrt{3-x}}$, so f is differentiable for all $x < 3$. Now all such x are interior points of the domain of f, so by Theorem 4.2, Local Extreme Values, if f has a local extrema at such an x value then f' must be 0 at that x value. We have found all such critical points of f, so there can be no other local extrema (and hence no other absolute extrema of f). Notice that we can make $f(x)$ large and positive by making x large and negative (so f has no absolute maximum; in particular we can make f larger than $f(12/5)$). The graph of f is something like (we have used red has marks to indicate critical points):



□

Exercise 4.1.72

Exercise 4.1.72. If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

Solution. YES! First, if $c = 0$ then $c = -c$ and we can (vacuously) say that f has a local maximum at $-c$. If f has a local maximum at $x = c \neq 0$, then by the definition of "local maximum" there is an open interval I containing c such that $f(x) \leq f(c)$ for all $x \in I$. Let $I = (a, b)$. Since f is hypothesized to be even, then $f(x) = f(-x)$ for all x in the domain of f. So for each $x \in (-b, -a)$, we have $-x \in (a, b) = I$, and for all such x we have $f(x) = f(-x) \leq f(c) = f(-c)$. That is, there is an open interval containing $-c$, namely $(-b, -a)$, such that for all $x \in (-b, -a)$ we have $f(x) \leq f(-c)$. Therefore, f has a $\boxed{\text{local maximum value at } x = -c}$. □