Chapter 4. Applications of Derivatives
4.1. Extreme Values of Functions on Closed Intervals—Examples and Proofs
<table>
<thead>
<tr>
<th></th>
<th>Exercise 4.1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Exercise 4.1.4</td>
</tr>
<tr>
<td>3</td>
<td>Theorem 4.2. Local Extreme Values</td>
</tr>
<tr>
<td>4</td>
<td>Exercise 4.1.24</td>
</tr>
<tr>
<td>5</td>
<td>Exercise 4.1.44</td>
</tr>
<tr>
<td>6</td>
<td>Exercise 4.1.60</td>
</tr>
<tr>
<td>7</td>
<td>Exercise 4.1.72. Even Functions</td>
</tr>
</tbody>
</table>
Exercise 4.1.2. Determine from the graph whether $f$ has any absolute extreme values on $[a, b]$:

Solution. First, $f$ is continuous on $[a, b]$ so by Theorem 4.1, The Extreme-Value Theorem for Continuous Functions, it has both an absolute maximum and absolute minimum. From the graph, we see that $f$ has an absolute maximum of $f(c)$ and an absolute minimum of $f(b)$. \qed
Exercise 4.1.2. Determine from the graph whether $f$ has any absolute extreme values on $[a, b]$:

Solution. First, $f$ is continuous on $[a, b]$ so by Theorem 4.1, The Extreme-Value Theorem for Continuous Functions, it has both an absolute maximum and absolute minimum. From the graph, we see that $f$ has an absolute maximum of $f(c)$ and an absolute minimum of $f(b)$. □
Exercise 4.1.4. Determine from the graph whether $h$ has any absolute extreme values on $[a, b]$:

Solution. First, $h$ is not defined on $[a, b]$, since $h$ is not defined at $x = a$ nor at $x = b$. In addition, $h$ is not defined at $x = c$. So Theorem 4.1 does not apply.
Exercise 4.1.4. Determine from the graph whether \( h \) has any absolute extreme values on \([a, b]\):

Solution. First, \( h \) is not defined on \([a, b]\), since \( h \) is not defined at \( x = a \) nor at \( x = b \). In addition, \( h \) is not defined at \( x = c \). So Theorem 4.1 does not apply. In fact, \( h \) has neither an absolute maximum nor an absolute minimum.
Exercise 4.1.4. Determine from the graph whether $h$ has any absolute extreme values on $[a, b]$:

Solution. First, $h$ is not defined on $[a, b]$, since $h$ is not defined at $x = a$ nor at $x = b$. In addition, $h$ is not defined at $x = c$. So Theorem 4.1 does not apply. In fact, $h$ has neither an absolute maximum nor an absolute minimum.
Solution (continued). We see that 
\[ \lim_{x \to a^+} h(x) \] exists and is strictly greater than any value of \( h(x) \) for \( x \in (a, b) \), and 
\[ \lim_{x \to c} h(x) \] exists and is strictly less than any value of \( h(x) \) for \( x \in (a, b) \).
So these values are upper and lower bounds on the values of \( h \), but neither value is attained by \( h \) on \((a, b)\). In fact, values of \( h \) can be made arbitrarily close to both of these values (by making \( x \) sufficiently close to \( a \) and greater than \( a \) for the upper bound 
\[ \lim_{x \to a^+} h(x) \], and by making \( x \) sufficiently close to \( c \) for the lower bound 
\[ \lim_{x \to c} h(x) \]). This is related to the idea that there is not a least positive real number (nor a greatest negative real number); remember that 0 is neither positive nor negative... because it is too busy being 0! □
Solution (continued). We see that
\[ \lim_{x \to a^+} h(x) \] exists and is strictly
greater than any value of \( h(x) \) for \( x \in (a, b) \),
and \( \lim_{x \to c} h(x) \) exists and is strictly
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\( \lim_{x \to c} h(x) \)). This is related to the idea that there is not a least positive
real number (nor a greatest negative real number); remember that 0 is
neither positive nor negative. . . because it is too busy being 0! □
Theorem 4.2. Local Extreme Values.

If a function $f$ has a local maximum value or a local minimum value at an interior point $c$ of its domain, and if $f'$ exists at $c$, then $f'(c) = 0$.

**Proof.** Suppose that $f$ has a local maximum value at $x = c$, so that $f(x) - f(c) \leq 0$ for all values of $x$ in some open interval containing $c$. Since $c$ is an interior point of the domain of $f$, then $f'(c)$ is (by the alternative definition of the derivative; see Exercise 3.2.24)

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}. $$
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$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$  

Considering one-sided limits and the fact that $f(c)$ is a local maximum of $f$, we have $f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0$ since $f(x) - f(c) \leq 0$ and for $x \to c^+$ we have $x - c > 0$, and $f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$ since $f(x) - f(c) \leq 0$ and for $x \to c^-$ we have $x - c < 0$.  

Local maximum value

Secant slopes $\geq 0$
(never negative)

Secant slopes $\leq 0$
(never positive)

$x$
Theorem 4.2. Local Extreme Values.

Theorem 4.2. If a function \( f \) has a local maximum value or a local minimum value at an interior point \( c \) of its domain, and if \( f' \) exists at \( c \), then \( f'(c) = 0 \).

Proof. Suppose that \( f \) has a local maximum value at \( x = c \), so that \( f(x) - f(c) \leq 0 \) for all values of \( x \) in some open interval containing \( c \). Since \( c \) is an interior point of the domain of \( f \), then \( f'(c) \) is (by the alternative definition of the derivative; see Exercise 3.2.24)

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.
\]

Considering one-sided limits and the fact that \( f(c) \) is a local maximum of \( f \), we have

\[
f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0
\]

since \( f(x) - f(c) \leq 0 \) and for \( x \to c^+ \) we have \( x - c > 0 \), and

\[
f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0
\]

since \( f(x) - f(c) \leq 0 \) and for \( x \to c^- \) we have \( x - c < 0 \).
Theorem 4.2. Local Extreme Values.

If a function $f$ has a local maximum value or a local minimum value at an interior point $c$ of its domain, and if $f'$ exists at $c$, then $f'(c) = 0$.

Proof (continued). Since the two-sided limit exists, then the one-sided limits must both exist and be the same by Theorem 2.6. ("Relation Between One-Sided and Two-Sided Limits"), so we must have $f'(c) = 0$.

The argument when $f$ has a local minimum value at $x = c$ (we then have $f(x) - f(c) \geq 0$ for all values of $x$ in some open interval containing $c$ and the inequalities in the one-sided limits are reversed) is similar.
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If a function $f$ has a local maximum value or a local minimum value at an interior point $c$ of its domain, and if $f'$ exists at $c$, then $f'(c) = 0$.

**Proof (continued).** Since the two-sided limit exists, then the one-sided limits must both exist and be the same by Theorem 2.6. ("Relation Between One-Sided and Two-Sided Limits"), so we must have $f'(c) = 0$.

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Exercise 4.1.24

**Exercise 4.1.24.** Find the absolute maximum and minimum values of $f(x) = 4 - x^3$ on the interval $[-2, 1]$. Then graph $y = f(x)$ and identify the points on the graph where the absolute extrema occur.

**Solution.** We follow the three steps just introduced.
Exercise 4.1.24. Find the absolute maximum and minimum values of $f(x) = 4 - x^3$ on the interval $[-2, 1]$. Then graph $y = f(x)$ and identify the points on the graph where the absolute extrema occur.

Solution. We follow the three steps just introduced. With $f(x) = 4 - x^3$, we have $f'(x) = -3x^2$ and for Step 1 we set $f'(x) = -3x^2 = 0$ and see that $x = 0$ is the only critical point.
Exercise 4.1.24. Find the absolute maximum and minimum values of \( f(x) = 4 - x^3 \) on the interval \([-2, 1]\). Then graph \( y = f(x) \) and identify the points on the graph where the absolute extrema occur.

Solution. We follow the three steps just introduced. With \( f(x) = 4 - x^3 \), we have \( f'(x) = -3x^2 \) and for Step 1 we set \( f'(x) = -3x^2 = 0 \) and see that \( x = 0 \) is the only critical point. For Step 2, we consider the values of \( f \) at the critical point \( x = 0 \) and the endpoints \( a = -2 \) and \( b = 1 \):

\[
\begin{array}{|c|c|c|c|}
\hline
x & -2 & 0 & 1 \\
\hline
f(x) & 4 - (-2)^3 = 12 & 4 - (0)^3 = 4 & 4 - (1)^3 = 3 \\
\hline
\end{array}
\]

By Step 3, the absolute maximum is 12 and occurs at \( x = -2 \), and the absolute minimum is 3 and occurs at \( x = 1 \).
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Exercise 4.1.24. Find the absolute maximum and minimum values of \( f(x) = 4 - x^3 \) on the interval \([-2, 1]\). Then graph \( y = f(x) \) and identify the points on the graph where the absolute extrema occur.

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<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>( 4 - (-2)^3 = 12 )</td>
<td>( 4 - (0)^3 = 4 )</td>
<td>( 4 - (1)^3 = 3 )</td>
</tr>
</tbody>
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By Step 3, the absolute maximum is 12 and occurs at \( x = -2 \), and the absolute minimum is 3 and occurs at \( x = 1 \).
Exercise 4.1.24. Find the absolute maximum and minimum values of \( f(x) = 4 - x^3 \) on the interval \([-2, 1]\). Then graph \( y = f(x) \) and identify the points on the graph where the absolute extrema occur.

**Solution.** We follow the three steps just introduced. With \( f(x) = 4 - x^3 \), we have \( f'(x) = -3x^2 \) and for Step 1 we set \( f'(x) = -3x^2 = 0 \) and see that \( x = 0 \) is the only critical point. For Step 2, we consider the values of \( f \) at the critical point \( x = 0 \) and the endpoints \( a = -2 \) and \( b = 1 \):

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<tr>
<td>( f(x) )</td>
<td>4 - (-2)^3 = 12</td>
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</tr>
</tbody>
</table>

By Step 3, the absolute maximum is 12 and occurs at \( x = -2 \), and the absolute minimum is 3 and occurs at \( x = 1 \).
Solution (continued). The graph is:

\[ y = 4 - x^3 \]

The graph includes points \((-2, 12)\) and \((1, 3)\).
Exercise 4.1.44

Exercise 4.1.44. Find the absolute maximum and minimum values of \( h(\theta) = 3\theta^{2/3} \) on the interval \([-27, 8]\).

Solution. We follow the three steps.
Exercise 4.1.44

Exercise 4.1.44. Find the absolute maximum and minimum values of \( h(\theta) = 3\theta^{2/3} \) on the interval \([-27, 8]\).

Solution. We follow the three steps. With \( h(\theta) = 3\theta^{2/3} \), we have
\[
h'(\theta) = 3(2/3)\theta^{-1/3} = \frac{2}{\sqrt[3]{\theta}}
\]
and for Step 1 we see that \( h' \) is never 0, but \( h' \) is undefined at \( \theta = 0 \). So \( \theta = 0 \) is the only critical point.

Exercise 4.1.44. Find the absolute maximum and minimum values of $h(\theta) = 3\theta^{2/3}$ on the interval $[-27, 8]$.

Solution. We follow the three steps. With $h(\theta) = 3\theta^{2/3}$, we have 

$$h'(\theta) = 3(2/3)\theta^{-1/3} = \frac{2}{\sqrt[3]{\theta}}$$

and for Step 1 we see that $h'$ is never 0, but $h'$ is undefined at $\theta = 0$. So $\theta = 0$ is the only critical point. For Step 2, we consider the values of $h$ at the critical point $\theta = 0$ and the endpoints $a = -27$ and $b = 8$:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$-27$</th>
<th>0</th>
<th>8</th>
</tr>
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<tbody>
<tr>
<td>$h(\theta)$</td>
<td>$3(-27)^{2/3} = 27$</td>
<td>$3(0)^{2/3} = 0$</td>
<td>$3(8)^{2/3} = 12$</td>
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**Solution.** We follow the three steps. With \( h(\theta) = 3\theta^{2/3} \), we have

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h'(\theta) = 3(2/3)\theta^{-1/3} = \frac{2}{\sqrt[3]{\theta}}
\]

and for Step 1 we see that \( h' \) is never 0, but \( h' \) is undefined at \( \theta = 0 \). So \( \theta = 0 \) is the only critical point. For Step 2, we consider the values of \( h \) at the critical point \( \theta = 0 \) and the endpoints \( a = -27 \) and \( b = 8 \):

<table>
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<td>( h(\theta) )</td>
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<td>( 3(8)^{2/3} = 12 )</td>
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By Step 3, the absolute maximum is 27 and occurs at \( \theta = -27 \), and the absolute minimum is 0 and occurs at \( \theta = 0 \). □
Exercise 4.1.44. Find the absolute maximum and minimum values of $h(\theta) = 3\theta^{2/3}$ on the interval $[-27, 8]$.

Solution. We follow the three steps. With $h(\theta) = 3\theta^{2/3}$, we have $h'(\theta) = 3(2/3)\theta^{-1/3} = \frac{2}{\sqrt[3]{\theta}}$ and for Step 1 we see that $h'$ is never 0, but $h'$ is undefined at $\theta = 0$. So $\theta = 0$ is the only critical point. For Step 2, we consider the values of $h$ at the critical point $\theta = 0$ and the endpoints $a = -27$ and $b = 8$:

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<tr>
<td>$h(\theta)$</td>
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By Step 3, the absolute maximum is 27 and occurs at $\theta = -27$, and the absolute minimum is 0 and occurs at $\theta = 0$. □
Exercise 4.1.60

**Exercise 4.1.60.** Find the critical points and domain endpoints for $y = f(x) = x^2 \sqrt{3 - x}$. Then find the value of the function at each of these points and identify extreme values (absolute and local).

**Solution.** First, notice that the domain of $f$ is $(-\infty, 3]$ (that is, $x \leq 3$ where $3 - x \geq 0$), so 3 is an endpoint of the domain. Also, $f$ is nonnegative. Since the domain is not an interval of the form $[a, b]$, we cannot precisely follow the three steps.

The critical points are $x = 0$ (because $f'(0) = 0$), $x = \frac{12}{5}$ (because $f'(\frac{12}{5}) = 0$), and $x = 3$ (because $x = 3$ is in the domain of $f$ but $f'$ is not defined at $x = 3$).
Exercise 4.1.60. Find the critical points and domain endpoints for
\( y = f(x) = x^2 \sqrt{3 - x} \). Then find the value of the function at each of
these points and identify extreme values (absolute and local).

Solution. First, notice that the domain of \( f \) is \((-\infty, 3]\) (that is, \( x \leq 3 \)
where \( 3 - x \geq 0 \)), so 3 is an endpoint of the domain. Also, \( f \) is
nonnegative. Since the domain is not an interval of the form \([a, b]\), we
cannot precisely follow the three steps. But we still need the critical points
of \( f(x) = x^2(3 - x)^{1/2} \) and so consider

\[
\begin{align*}
f'(x) &= [2x][(3 - x)^{1/2}) + (x^2)[(1/2)(3 - x)^{-1/2}[-1]] = \\
&= 2x\sqrt{3 - x} - \frac{x^2}{2\sqrt{3 - x}} = 2x\sqrt{3 - x} \left(\frac{2\sqrt{3 - x}}{2\sqrt{3 - x}}\right) - \frac{x^2}{2\sqrt{3 - x}} = \\
&= \frac{4x(3 - x) - x^2}{2\sqrt{3 - x}} = \frac{12x - 5x^2}{2\sqrt{3 - x}} = \frac{x(12 - 5x)}{2\sqrt{3 - x}}. \text{ The critical points are } x = 0 \\
&\quad \text{(because } f'(0) = 0) \quad \text{and } x = 3.
\end{align*}
\]

(because \( x = 3 \) is in the domain of \( f \) but \( f' \) is not defined at \( x = 3 \)).
Exercise 4.1.60. Find the critical points and domain endpoints for \( y = f(x) = x^2 \sqrt{3 - x} \). Then find the value of the function at each of these points and identify extreme values (absolute and local).

Solution. First, notice that the domain of \( f \) is \((-\infty, 3]\) (that is, \( x \leq 3 \) where \( 3 - x \geq 0 \)), so 3 is an endpoint of the domain. Also, \( f \) is nonnegative. Since the domain is not an interval of the form \([a, b]\), we cannot precisely follow the three steps. But we still need the critical points of \( f(x) = x^2(3 - x)^{1/2} \) and so consider

\[
    f'(x) = [2x][(3 - x)^{1/2}] + (x^2)[(1/2)(3 - x)^{-1/2}][-1] = 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}} = 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}} = 4(x(3-x) - x^2) = 12x - 5x^2 = \frac{x(12 - 5x)}{2\sqrt{3-x}}.
\]

The critical points are \( x = 0 \) (because \( f'(0) = 0 \)), \( x = 12/5 \) (because \( f'(12/5) = 0 \)), and \( x = 3 \) (because \( x = 3 \) is in the domain of \( f \) but \( f' \) is not defined at \( x = 3 \)).
Solution (continued). We consider the values of $f$ at the critical points and endpoint:

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
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<tr>
<td>$f(x)$</td>
<td>$(0)^2\sqrt{3 - (0)} = 0$</td>
<td>$(12/5)^2\sqrt{3 - 12/5} = (144/25)\sqrt{3/5}$</td>
<td>$(3)^2\sqrt{3 - (3)} = 0$</td>
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Since $f(x) \geq 0$ for all $x$ in its domain, then $f$ must have an absolute minimum at $x = 0$ and $x = 3$ of 0. Next, we claim that $f$ has a local maximum at $x = 12/5$. This is because $12/5$ is between 0 and 3, and $f(12/5) > f(0) = f(3)$; for if $f$ had a larger value than $f(12/5)$ for some $0 < x < 3$, then (since $f$ is differentiable for $0 < x < 3$) by Theorem 4.2, Local Extreme Values, $f$ would have another critical point between 0 and 3 where the derivative is 0, but there is no such point. So $f(12/5)$ must be the largest value of $f$ on the open interval $(0, 3)$ and hence $f$ has a local maximum at $x = 12/5$ of $(144/25)\sqrt{3/5}$. 
Solution (continued). We consider the values of $f$ at the critical points and endpoint:

<table>
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<td>$f(x)$</td>
<td>$(0)^2 \sqrt{3} - (0) = 0$</td>
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Since $f(x) \geq 0$ for all $x$ in its domain, then $f$ must have an absolute minimum at $x = 0$ and $x = 3$ of $0$. Next, we claim that $f$ has a local maximum at $x = 12/5$. This is because $12/5$ is between $0$ and $3$, and $f(12/5) > f(0) = f(3)$; for if $f$ had a larger value than $f(12/5)$ for some $0 < x < 3$, then (since $f$ is differentiable for $0 < x < 3$) by Theorem 4.2, Local Extreme Values, $f$ would have another critical point between $0$ and $3$ where the derivative is $0$, but there is no such point. So $f(12/5)$ must be the largest value of $f$ on the open interval $(0, 3)$ and hence $f$ has a local maximum at $x = 12/5$ of $(144/25)\sqrt{3}/5$. 
Solution (continued). As shown above, \( f'(x) = \frac{x(12 - 5x)}{2\sqrt{3 - x}} \), so \( f \) is differentiable for all \( x < 3 \). Now all such \( x \) are interior points of the domain of \( f \), so by Theorem 4.2, Local Extreme Values, if \( f \) has a local extrema at such an \( x \) value then \( f' \) must be 0 at that \( x \) value. We have found all such critical points of \( f \), so there can be no other local extrema (and hence no other absolute extrema of \( f \)). Notice that we can make \( f(x) \) large and positive by making \( x \) large and negative (so \( f \) has no absolute maximum); in particular, we can make \( f \) larger than \( f(12/5) \).
Solution (continued). As shown above, $f'(x) = \frac{x(12 - 5x)}{2\sqrt{3-x}}$, so $f$ is differentiable for all $x < 3$. Now all such $x$ are interior points of the domain of $f$, so by Theorem 4.2, Local Extreme Values, if $f$ has a local extrema at such an $x$ value then $f'$ must be 0 at that $x$ value. We have found all such critical points of $f$, so there can be no other local extrema (and hence no other absolute extrema of $f$). Notice that we can make $f(x)$ large and positive by making $x$ large and negative (so $f$ has no absolute maximum); in particular, we can make $f$ larger than $f(12/5)$.

The graph of $f$ is something like (we have used red has marks to indicate critical points):
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Exercise 4.1.72

Exercise 4.1.72. If an even function \( f(x) \) has a local maximum value at \( x = c \), can anything be said about the value of \( f \) at \( x = -c \)? Give reasons for your answer.

Solution. YES! First, if \( c = 0 \) then \( c = -c \) and we can (vacuously) say that \( f \) has a local maximum at \( -c \). If \( f \) has a local maximum at \( x = c \neq 0 \), then by the definition of “local maximum” there is an open interval \( I \) containing \( c \) such that \( f(x) \leq f(c) \) for all \( x \in I \). Let \( I = (a, b) \).
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Since \( f \) is hypothesized to be even, then \( f(x) = f(-x) \) for all \( x \) in the domain of \( f \). So for each \( x \in (-b, -a) \), we have \( -x \in (a, b) = I \), and for all such \( x \) we have \( f(x) = f(-x) \leq f(c) = f(-c) \). That is, there is an open interval containing \( -c \), namely \( (-b, -a) \), such that for all \( x \in (-b, -a) \) we have \( f(x) \leq f(-c) \). Therefore, \( f \) has a local maximum value at \( x = -c \). □
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