Chapter 4. Applications of Derivatives
4.4. Concavity and Curve Sketching—Examples and Proofs

Exercise 4.4.2

Consider $f(x) = x^4/4 - 2x^2 + 4$. Identify the inflection points and local maxima and minima of $f$ and identify the intervals on which the function is concave up and concave down.

Solution. First, $f'(x) = x^3 - 4x = x(x^2 - 4) = x^2(x + 2)(x - 2)$ and we see that $-2$, $0$, and $2$ are critical points since $f'$ is 0 at these points. Next, $f''(x) = 3x^2 - 4$ so that $x = \pm\sqrt{4/3} = \pm2/\sqrt{3}$ are potential points of inflection.

Solution (continued). As in the previous section, since $f''(x) = 3x^2 - 4$ is a polynomial (and so is continuous by Theorem 2.5.A) then by the Intermediate Value Theorem (Theorem 2.11) the only way $f''$ can change sign as $x$ increases is for $f''$ to take on the value 0. That is, $f''$ has the same sign on the intervals $(-\infty, -2/\sqrt{3})$, $(-2/\sqrt{3}, 2/\sqrt{3})$, and $(2/\sqrt{3}, \infty)$. So we use test values from these intervals to determine the sign of $f''$ throughout these intervals.

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -2/\sqrt{3})$</th>
<th>$(-2/\sqrt{3}, 2/\sqrt{3})$</th>
<th>$(2/\sqrt{3}, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $k$</td>
<td>-2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$f''(k)$</td>
<td>$3(-2)^2 - 4 = 8$</td>
<td>$3(0)^2 - 4 = -4$</td>
<td>$3(2)^2 - 4 = 8$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>CU</td>
<td>CD</td>
<td>CU</td>
</tr>
</tbody>
</table>

Here, the concavity is given by the Second Derivative Test for Concavity (Theorem 4.4.A).

Exercise 4.4.2 (continued 2)

Solution (continued). ... 

We are given the graph of $f$, so we see that it has a local maximum of $f(0) = (0)^4/4 - 2(0)^2 + 4 = 4$ and a local minimum of $f(-2) = f(2) = (2)^4/4 - 2(2)^2 + 4 = 0$. □
Theorem 4.5. Second Derivative Test for Local Extrema.
Suppose $f''$ is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, a local minimum, or neither.

**Proof.** (1) If $f'' < 0$, then $f'' < 0$ on some open interval $I$ containing the point $c$, since $f''$ is continuous (by Exercise 2.570). Therefore by Corollary 4.3 ("The First Derivative Test for Increasing and Decreasing"), $f'$ is decreasing on $I$. Since $f'(c) = 0$, the sign of $f'$ changes from positive to negative as $x$ increases through the value $c$, and so $f$ has a local maximum at $x = c$ by Theorem 4.3.A(2), “First Derivative Test for Local Extrema,” as claimed.

**Theorem 4.5 (continued 2)**

**Theorem 4.5. Second Derivative Test for Local Extrema.**
Suppose $f''$ is continuous on an open interval that contains $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, a local minimum, or neither.

**Proof.** (3) We establish by this by giving examples. Consider $f_1(x) = x^4$, $f_2(x) = -x^4$, and $f_3(x) = x^3$. We have $f'_1(0) = f_2'(0) = f_3'(0) = 0$ (so we take $c = 0$), and $f_1''(0) = f_2''(0) = f_3''(0) = 0$. But $f_1(x) = x^4$ has a local minimum at $x = 0$, $f_2(x) = -x^4$ has a local maximum at $x = 0$, and $f_3(x) = x^3$ has neither a maximum nor a minimum at $x = 0$. So, as claimed, the test fails (is “inconclusive”).

Exercise 4.4.12

**Exercise 4.4.12.** Consider $y = f(x) = x(6 - 2x)^2$. Identify the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$.

**Solution.** First, $f'(x) = [1](6 - 2x)^2 + (x)[2(6 - 2x)] = (6 - 2x)(6 - 2x) = (6 - 2x)(6 - 2x) - 4x = (6 - 2x)(6 - 2x)$ so that $x = 1$ and $x = 3$ are critical points since $f'$ is 0 at these points. Next $f''(x) = [2][6 - 6x] + (6 - 2x)[6 - 6x] = -12 + 12x - 36 + 12 = -48 + 24x$, so $x = 2$ is a potential point of inflection. As above, since $f''$ is continuous then we test the sign of $f''$ as:

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $k$</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$f''(k)$</td>
<td>$-48 + 24(1) = -24$</td>
<td>$-48 + 24(3) = 24$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(x)$</td>
<td>CD</td>
<td>CU</td>
</tr>
</tbody>
</table>
Exercise 4.4.12 (continued 1)

Solution (continued). So \( f \) does in fact change concavity at \( x = 2 \). Notice \( f(2) = (2)(6 - 2(2))^2 = 8 \) so the point of inflection is \((2, 8)\). We used the critical points as test values above, so we see by the Second Derivative Test for Local Extrema (Theorem 4.5) that \( f \) has a local maximum at \( x = 1 \) of \( f(1) = (1)(6 - 2(1))^2 = 16 \) and \( f \) has a local minimum at \( x = 3 \) of \( f(3) = (3)(6 - 2(3))^2 = 0 \). The coordinates of the local maximum point is \((1, 16)\) and the coordinates of the local minimum point is \((3, 0)\).

To graph \( y = f(x) \), we plot each extreme point and the point of inflection. We use little horizontal hash marks “—” through the extreme points (since tangent lines are horizontal there) and we use a “X” to indicate a point of inflection. We also plot the \( x \)-intercepts \((0, 0)\) and \((3, 0)\), and the \( y \)-intercept \((0, 0)\). Finally, we flesh out the graph in a way that reflects the known concavity.

Exercise 4.4.104

Exercise 4.4.104. Sketch a smooth connected curve \( y = f(x) \) with:
\[
f(-2) = 8, \; f(0) = 4, \; f(2) = 0, \; f'(x) > 0 \text{ for } |x| > 2, \; f'(2) = f'(-2) = 0, \; f''(x) < 0 \text{ for } |x| < 2, \; f''(x) < 0 \text{ for } x < 0, \; \text{ and } f''(x) > 0 \text{ for } x > 0.
\]
Indicate points where \( f' \) is 0 with horizontal hash marks and indicate points of inflection with X’s.

Solution. Since \( f'(x) > 0 \) for \( |x| > 2 \) and \( f'(x) < 0 \) for \( |x| < 2 \), then by The First Derivative Test for Increasing and Decreasing (Corollary 4.3) \( f \) is INC on \((-\infty, -2) \cup (2, \infty)\) and \( f \) is DEC on \((-2, 2)\). Since \( f''(x) < 0 \) for \( x < 0 \) and \( f''(x) > 0 \) for \( x > 0 \), the by the Second Derivative Test for Concavity (Theorem 4.4.A) \( f \) is CU on \((0, \infty)\) and \( f \) is CD on \((-\infty, 0)\). We combine this information in a table:

<table>
<thead>
<tr>
<th>interval</th>
<th>((-\infty, -2))</th>
<th>((-2, 0))</th>
<th>((0, 2))</th>
<th>((2, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>\text{+}</td>
<td>\text{–}</td>
<td>\text{–}</td>
<td>\text{+}</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>\text{INC, CD}</td>
<td>\text{DEC, CD}</td>
<td>\text{DEC, CU}</td>
<td>\text{INC, CU}</td>
</tr>
</tbody>
</table>

Notice that \((0, f(0)) = (0, 4)\) is a point of inflection.
Exercise 4.4.42. Consider $y = f(x) = \sqrt[3]{x^3 + 1}$. Identify the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$. Indicate points where $f'$ is 0 with horizontal hash marks and indicate points of inflection with X's.

**Solution.** First, $f(x) = (x^3 + 1)^{1/3}$ and so

$$f'(x) = (1/3)(x^3 + 1)^{-2/3}[3x^2] = x^2(x^3 + 1)^{-2/3} = \frac{x^2}{(x^3 + 1)^{2/3}},$$

so $x = 0$ is a critical point since $f'(0) = 0$ and $x = -1$ is a critical point since $x = -1$ is in the domain of $f$ but $f'$ is undefined at $x = -1$. Next

$$f''(x) = \frac{2x[(x^3 + 1)^{-2/3} - (x^2)((-2/3)(x^3 + 1)^{-5/3}[3x^2])}{(x^3 + 1)^{2/3}} = \frac{2x(x^3 + 1) - 2x^4}{(x^3 + 1)^{5/3}} = \frac{2x}{(x^3 + 1)^{5/3}},$$

so $f$ has a potential point of inflection at $x = 0$ and at $x = -1$ (notice that $f''$ is undefined at $x = -1$, but we could show that $y = f(x)$ has a vertical tangent at $x = -1$).

Exercise 4.4.42 (continued 2)

**Solution (continued).** Since $f'(0) = 0$, $f'$ is undefined at $x = -1$, $f(-1) = 0$, $f(0) = 1$, and

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -1)$</th>
<th>$(-1, 0)$</th>
<th>$(0, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>INC, CU</td>
<td>INC, CD</td>
<td>INC, CU</td>
</tr>
</tbody>
</table>

then the graph of $y = f(x) = \sqrt[3]{x^3 + 1}$ is:

Exercise 4.4.42 (continued 1)

**Solution (continued).** We find the signs of $f''(x) = x^2/(x^3 + 1)^{2/3}$ and $f''(x) = 2x/(x^3 + 1)^{5/3}$ over the appropriate intervals:

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -1)$</th>
<th>$(1/2, 1)$</th>
<th>$(0, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $k$</td>
<td>$-2$</td>
<td>$-1/2$</td>
<td>$1$</td>
</tr>
<tr>
<td>$f'(k)$</td>
<td>$-2/(1-2)^{1/3}$</td>
<td>$-1/2$</td>
<td>$(1/2)^{-1}$</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>$-1/(x-1)^{1/3}$</td>
<td>$-1/(x-1)^{1/3}$</td>
<td>$(x+1)^{1/3}$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$2x/(x^3 + 1)^{5/3}$</td>
<td>$2x/(x^3 + 1)^{5/3}$</td>
<td>$2x/(x^3 + 1)^{5/3}$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>$INC, CU$</td>
<td>$INC, CD$</td>
<td>$INC, CU$</td>
</tr>
</tbody>
</table>

Since $f$ is always increasing then it has no local maximum nor local minimum (by the First Derivative Test for Local Extrema, Theorem 4.3.A(3)). Notice that $f$ changes concavity at $x = -1$ and $x = 0$, so the points of inflection are $(-1, f(-1)) = (-1, 0)$ and $(0, f(0)) = (0, 1)$.

Exercise 4.4.54

**Exercise 4.4.54.** Consider $y = f(x) = xe^{-x}$. Identity the coordinates of any local and absolute extreme points and inflection points. Graph $y = f(x)$. Indicate points where $f'$ is 0 with horizontal hash marks and indicate points of inflection with X's.

**Solution.** First $f'(x) = [1(e^{-x}) + x(e^{-x} = e^{-x}(1-x)$, so $x = 1$ is a critical point since $f'(1) = 0$. Next $f''(x) = [e^{-x}(1-x)](1-x) + (e^{-x})(1-x-1) = -e^{-x}(2-x)$ so $x = 2$ is a potential point of inflection. We perform a sign test on $f'$ and $f''$:

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, 1)$</th>
<th>$(1, 2)$</th>
<th>$(2, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $k$</td>
<td>$0$</td>
<td>$3/2$</td>
<td>$3$</td>
</tr>
<tr>
<td>$f'(k)$</td>
<td>$e^{-0}(1-0))$</td>
<td>$e^{-3/2}(1-3/2)$</td>
<td>$e^{-3}(1-3)$</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>$+1$</td>
<td>$-1/2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$-e^{-3}(2-0)$</td>
<td>$-e^{-3}(2-3/2)$</td>
<td>$-e^{-3}(2-3)$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>$-2$</td>
<td>$+1$</td>
<td>$+1$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>INC, CD</td>
<td>DEC, CD</td>
<td>DEC, CU</td>
</tr>
</tbody>
</table>
Exercise 4.4.74

**Exercise 4.4.74.** Let \( y = f(x) \) be a continuous function with \( y'(t) = \sin t \) for \( t \in [0, 2\pi] \). Find \( y'' \) and then use Steps 2–4 of the graphing procedure to sketch the general shape of the graph of \( f \). Indicate points where \( f' \) is 0 with horizontal hash marks.

**Solution.** First, if \( y'(t) = \sin t \) then \( y''(t) = \cos t \).

1. We have \( y' \) and \( y'' \) above.

2. Since \( y'(t) = \sin t \) then the critical points of \( y \) for \( t \in [0, 2\pi] \) are \( t = 0, t = \pi, \) and \( t = 2\pi \), since \( y' \) is 0 at each of these.

3. We perform a sign test on \( y'(t) = \sin t \):

<table>
<thead>
<tr>
<th>interval</th>
<th>((0, \pi))</th>
<th>((\pi, 2\pi))</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value ( k )</td>
<td>( \pi/4 )</td>
<td>( 5\pi/4 )</td>
</tr>
<tr>
<td>( f'(k) )</td>
<td>( \sin \pi/4 = \sqrt{2}/2 )</td>
<td>( \sin 5\pi/4 = -\sqrt{2}/2 )</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>( f(x) )</td>
<td>INC</td>
<td>DEC</td>
</tr>
</tbody>
</table>

Exercise 4.4.92

**Exercise 4.4.92.** Graph the rational function \( y = f(x) = \frac{x^2 - 4}{x^2 - 2} \). Use all the steps in the graphing procedure. Indicate points where \( f' \) is 0 with horizontal hash marks and indicate points of inflection with X’s.

**Solution.** We throw all of our graphing knowledge at this one!

1. With \( y = f(x) = \frac{x^2 - 4}{x^2 - 2} \), the domain is all \( x \in \mathbb{R} \) except \( x = \pm \sqrt{2} \) (since the denominator is 0 there). That is, the domain is \((-\infty, -\sqrt{2}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{2}, \infty)\). Notice that \( f(-x) = \frac{(-x)^2 - 4}{(-x)^2 - 2} = \frac{x^2 - 4}{x^2 - 2} = f(x) \), so \( f \) is an even function and hence symmetric with respect to the y-axis.
Exercise 4.4.92 (continued 1)

Solution (continued). (2) We have
\[ y' = \frac{4x}{(x^2 - 2)^2} \text{ and } y'' = \frac{4(3x^2 + 2)}{(x^2 - 2)^3}. \]

(3) We see from \( y' = \frac{4x}{(x^2 - 2)^2} \) that \( x = 0 \) is the only critical point (since \( \pm \sqrt{2} \) are not in the domain of \( f \), and \( f'(0) = 0 \). Notice \( f(0) = \frac{0^2 - 4}{0^2 - 2} = 2 \).

Exercise 4.4.92 (continued 2)

Solution (continued). (4) We perform a sign test on \( y' \) by removing the critical point from the domain of \( y' \):

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{interval} & (\infty, -\sqrt{2}) & (-\sqrt{2}, 0) & (0, \sqrt{2}) & (\sqrt{2}, \infty) \\
\hline
\text{test value } k & -2 & -1 & 1 & 2 \\
\hline
f''(k) & \frac{-4(\sqrt{2}+2)}{(2\sqrt{2}-1)^2} & \frac{4(\sqrt{2}+1)}{(2\sqrt{2}-1)^2} & \frac{4(\sqrt{2}+1)}{(2\sqrt{2}-1)^2} & \frac{4(\sqrt{2}+2)}{(2\sqrt{2}-1)^2} \\
\hline
f'(x) & \text{DEC} & \text{DEC} & \text{INC} & \text{INC} \\
\hline
f(x) & & & & \\
\hline
\end{array}
\]

So \( y \) is decreasing on \((-\infty, -\sqrt{2}) \cup (-\sqrt{2}, 0) \) and \( y \) is increasing on \((0, \sqrt{2}) \cup (\sqrt{2}, \infty) \).

(5) We see from \( y'' = \frac{4(3x^2 + 2)}{(x^2 - 2)^3} \) that \( y \) has no potential points of inflection (since the numerator is never 0 and the denominator is never 0 at points in the domain of \( y \)). So we perform a sign test on \( y'' \) on the domain of \( y \).

Exercise 4.4.92 (continued 3)

Solution (continued).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{interval} & (-\infty, -\sqrt{2}) & (-\sqrt{2}, \sqrt{2}) & (\sqrt{2}, \infty) \\
\hline
\text{test value } k & -2 & 0 & 2 \\
\hline
f''(k) & -\frac{4(\sqrt{2}+2)}{(2\sqrt{2}-1)^2} & -\frac{4(\sqrt{2}+1)}{(2\sqrt{2}-1)^2} & -\frac{4(\sqrt{2}+1)}{(2\sqrt{2}-1)^2} \\
\hline
f'(x) & - & + & - \\
\hline
f(x) & CD & CU & CD \\
\hline
\end{array}
\]

So \( y \) is \( \text{CU} \) on \((-\sqrt{2}, \sqrt{2}) \) and \( y \) is \( \text{CD} \) on \((-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty) \).

(6) Now for asymptotes. Notice
\[
\lim_{x \to \pm \infty} \frac{x^2 - 4}{x^2 - 2} = \lim_{x \to \pm \infty} \frac{x^2 - 4}{x^2 - 2} \frac{1}{x^2} = \lim_{x \to \pm \infty} \frac{x^2 - 4}{x^2 - 2} = 1.
\]

So \( y = 1 \) is a horizontal asymptote.

Exercise 4.4.92 (continued 4)

Solution (continued). By Dr. Bob’s Infinite Limits Theorem,
\[
f(x) = \frac{x^2 - 4}{x^2 - 2} \Rightarrow \lim_{x \to \pm \sqrt{2}} f(x) = \pm \infty
\]
and so \( f \) has vertical asymptotes at \( x = \pm \sqrt{2} \). We consider the four sign diagrams: (1) For \( x \to -\sqrt{2}^- \) (so that \( x \) is “close to” \(-\sqrt{2} \) and less than \(-\sqrt{2} \)) we have \( x^2 - 4 \Rightarrow \frac{-}{-} \Rightarrow (-) \Rightarrow - \), (2) for \( x \to -\sqrt{2}^+ \) we have \( x^2 - 4 \Rightarrow \frac{-}{+} \Rightarrow (+) \Rightarrow + \), (3) for \( x \to \sqrt{2}^- \) we have \( x^2 - 4 \Rightarrow \frac{-}{-} \Rightarrow (-) \Rightarrow + \), and (4) for \( x \to \sqrt{2}^+ \) we have \( x^2 - 4 \Rightarrow \frac{-}{+} \Rightarrow (+) \Rightarrow + \). So \( \lim_{x \to \sqrt{2}^-} f(x) = -\infty, \lim_{x \to \sqrt{2}^+} f(x) = \infty, \lim_{x \to \sqrt{2}^-} f(x) = \infty, \text{ and } \lim_{x \to \sqrt{2}^+} f(x) = -\infty.\)
Exercise 4.4.92 (continued 5)

Solution (continued).

(7) We have:

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -\sqrt{2})$</th>
<th>$(-\sqrt{2}, 0)$</th>
<th>$(0, \sqrt{2})$</th>
<th>$(\sqrt{2}, \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>DEC, CD</td>
<td>DEC, CU</td>
<td>INC, CU</td>
<td>INC, CD</td>
</tr>
</tbody>
</table>

Since $y = f(x) = \frac{x^2 - 4}{x^2 - 2}$, then the y-intercepts are $x = \pm 2$. We have $f(0) = 2$ from above. So...

Exercise 4.4.122

Exercise 4.4.122. Parabolas.

(a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, where $a \neq 0$. (b) When is the parabola concave up? Concave down? Give reasons for your answer.

Solution. We have $y' = 2ax + b$ and $y'' = 2a$.

The vertex of a parabola $y = ax^2 + bx + c$ is an absolute extreme of the function $f(x) = ax^2 + bx + c$, and hence a local extreme value. So by Theorem 4.2, “Local Extreme Values,” the vertex occurs at a critical point of $f$. Since $f'(x) = 2ax + b$ then the only critical point is $x = -b/(2a)$.

Since $f\left(-\frac{b}{2a}\right) = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = -\frac{b^2}{4a} - \frac{b^2}{2a} + c =\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a} = \frac{-b^2 + 4ac}{4a}$ So the coordinates of the vertex is $(-b/(2a), f(-b/(2a))) = (-b/a, (-b^2 + 4ac)/(4a))$.

Exercise 4.4.122 (continued)

Exercise 4.4.124

Exercise 4.4.124. Parabolas.

(a) Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c$, where $a \neq 0$. (b) When is the parabola concave up? Concave down? Give reasons for your answer.

Solution (continued). (b) Since $y'' = f''(x) = 2a$ then by the Second Derivative Test for Concavity (Theorem 4.4.A), the parabola is concave up everywhere when $a > 0$ and the parabola is concave down everywhere when $a < 0$.

Note. Notice that the second degree polynomial function $f(x) = ax^2 + bx + c$ has no point of inflection.


What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$? Give reasons for your answer.

Solution. We have $y' = 3ax^2 + 2bx + c$ and $y'' = 6ax + 2b$. With $y = f(x)$ we have $f''(-b/(3a)) = 0$ then $-b/(3a)$ is a potential point of inflection. So we perform a sign test on $f''(x)$:

<table>
<thead>
<tr>
<th>interval</th>
<th>$(-\infty, -b/(3a))$</th>
<th>$(-b/(3a), \infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $k$</td>
<td>$-b/(3a) - 1$</td>
<td>$-b/(3a) + 1$</td>
</tr>
<tr>
<td>$f''(k)$</td>
<td>$6a(-b/(3a) - 1) + 2b = -6a$</td>
<td>$6a(-b/(3a) + 1) + 2b = 6a$</td>
</tr>
</tbody>
</table>

Since $a \neq 0$ by hypothesis, then by the Second Derivative Test for Concavity (Theorem 4.4.A) $f$ changes concavity at $x = -b/(3a)$. So there is only one inflection point and, since...
What can you say about the inflection points of a cubic curve
\( y = ax^3 + bx^2 + cx + d \), where \( a \neq 0 \)? Give reasons for your answer.

Solution (continued). So there is only one inflection point and, since

\[
f \left( -\frac{b}{3a} \right) = a \left( -\frac{b}{3a} \right)^3 + b \left( -\frac{b}{3a} \right)^2 + c \left( -\frac{b}{3a} \right) + c
\]

\[
= \frac{-b^3}{27a^2} + \frac{b^3}{9a^2} + \frac{-bc}{3a} + c = \frac{-b^3}{27a^2} + \frac{3b^3}{27a^2} + \frac{-9abc}{27a^2} + \frac{27a^2c}{27a^2}
\]

\[
= \frac{2b^3 - 9abc + 27a^2c}{27a^2},
\]

the inflection point is \( \left( -\frac{b}{3a}, \frac{2b^3 - 9abc + 27a^2c}{27a^2} \right) \). \( \square \)