Chapter 4. Applications of Derivatives

4.5. Indeterminate Forms and L’Hôpital’s Rule—Examples and Proofs

Exercise 4.5.16

Exercise 4.5.16. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate $\lim_{x \to 0} \frac{\sin x - x}{x^3}$. Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Solution. With $f(x) = \sin x - x$ and $g(x) = x^3$ and $a = 0$, we have $f(a) = g(a) = 0$, $f'(x) = \cos x - 1$, and $g'(x) = 3x^2$. So the hypotheses of l’Hôpital’s Rule hold, but $f'(a)/g'(a)$ does not exist since $g'(0) = 0$. However, $\lim_{x \to 0} f'(x)/g'(x)$ is itself of the $0/0$ indeterminate form so that we may attempt to apply l’Hôpital’s Rule to that (or maybe even $\lim_{x \to 0} f''(x)/g''(x)$). We then have

\[
\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = \frac{0}{0}
\]

\[
\lim_{x \to 0} \frac{-\cos x}{6} = \frac{-\cos(0)}{6} = \frac{-1}{6}.
\]

Exercise 4.5.38

Exercise 4.5.38. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate $\lim_{x \to 0^+} (\ln x - \ln \sin x)$. Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Solution. First, we rewrite the function $\ln x - \ln \sin x$ as $\ln \frac{x}{\sin x}$. Then

\[
\lim_{x \to 0^+} (\ln x - \ln \sin x) = \lim_{x \to 0^+} \frac{x}{\sin x}
\]

\[
= \ln \left( \lim_{x \to 0^+} \frac{x}{\sin x} \right)
\]

\[
= \ln \left( \lim_{x \to 0^+} \frac{1}{\cos x} \right)
\]

\[
= \ln \left( \frac{1}{\cos(0)} \right)
\]

\[
= \ln(1) = 0.
\]
Exercise 4.5.46

Exercise 4.5.46. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate \( \lim_{x \to \infty} x^2 e^{-x} \). Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Proof. With \( f(x) = e^{-x} \) and \( g(x) = x^2 \) we have 
\[ \lim_{x \to \infty} g(x) = \lim_{x \to \infty} x^2 = \infty \text{ and} \]
\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} e^x = 0 \] by Example 2.6.5 (where we have replaced \( x \) with \(-x\)). So \( \lim_{x \to \infty} x^2 e^{-x} \) is of the \( 0 \cdot \infty \) indeterminate form. We rewrite the function \( x^2 e^{-x} \) as \( x^2 / e^x \) and note that \( \lim_{x \to \infty} x^2 = \infty \) and \( \lim_{x \to \infty} e^x = \infty \), so that \( \lim_{x \to \infty} x^2 / e^x \) is of the \( \infty / \infty \) indeterminate form. So we have by Theorem 4.5.A, “L’Hôpital’s Rule for \( \infty / \infty \) Indeterminate Forms,” that
\[
\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = \frac{2 \lim_{x \to \infty} e^{-x}}{x} = 2(0) = 0 \] by Example 2.6.5. \( \square \)

Exercise 4.5.40

Exercise 4.5.40. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate \( \lim_{x \to 0^+} \left( \frac{3x + 1}{x} - \frac{1}{\sin x} \right) \). Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Solution. With \( f(x) = (3x + 1)/x \) and \( g(x) = 1/\sin x = \csc x \) we have 
\( \lim_{x \to 0^+} (3x + 1)/x = \infty \) (by Dr. Bob’s Infinite Limits Theorem) and \( \lim_{x \to 0^+} \csc x = \infty \) (see the graph of \( y = \csc x \)), so 
\( \lim_{x \to 0^+} \left( \frac{3x + 1}{x} - \frac{1}{\sin x} \right) \) is of the \( \infty - \infty \) indeterminate form. So we get a common denominator as follows
\[
\lim_{x \to 0^+} \left( \frac{3x + 1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \left( \frac{(3x + 1) \sin x - x}{x \sin x} - \frac{1}{x \sin x} \right)
= \lim_{x \to 0^+} \frac{(3x + 1) \sin x - x}{2 \cos x - x \sin x} \geq \lim_{x \to 0^+} \frac{[3(\sin x) + (3x + 1)(\cos x) - 1]}{[3(\sin x) + (3x + 1)(\cos x) - 1]}
= \lim_{x \to 0^+} \frac{6 \cos x - (3x + 1) \sin x}{6 \cos x - (3(0) + 1) \sin(0)} = \frac{6}{2} = 3 \square
\]

Exercise 4.5.32

Exercise 4.5.32. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate \( \lim_{x \to \infty} \frac{\log_2 x}{\log_3 (x + 3)} \). Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Solution. With \( f(x) = \log_2 x \) and \( g(x) = \log_3 (x + 3) \), we have 
\( \lim_{x \to \infty} \log_2 x = \lim_{x \to \infty} \log_3 (x + 3) = \infty \), so 
\( \lim_{x \to \infty} \frac{\log_2 x}{\log_3 (x + 3)} \) is of the \( \infty / \infty \) indeterminate form. So by Theorem 4.5.A, “L’Hôpital’s Rule for \( \infty / \infty \) Indeterminate Forms,”
\[
\lim_{x \to \infty} \frac{\log_2 x}{\log_3 (x + 3)} = \lim_{x \to \infty} \frac{(1/\ln 2)(1/x)}{(1/\ln 3)(1/(x + 3))}
= \frac{\ln 3}{\ln 2} \lim_{x \to \infty} \frac{x + 3}{x} = \frac{\ln 3}{\ln 2} \lim_{x \to \infty} \frac{1}{1} = \frac{\ln 3}{\ln 2} \square
\]

Exercise 4.5.40 (continued)

Exercise 4.5.40. Use l’Hôpital’s Rule (Theorem 4.6) to evaluate \( \lim_{x \to 0^+} \left( \frac{3x + 1}{x} - \frac{1}{\sin x} \right) \). Write the indeterminate form over the equal sign when you use l’Hôpital’s Rule.

Solution (continued). . .
\[
= \lim_{x \to 0^+} \frac{3(\sin x) + (3x + 1)(\cos x) - 1}{[3(\sin x) + (3x + 1)(\cos x) - 1]} = \lim_{x \to 0^+} \frac{3 \sin x + (3x + 1) \cos x - 1}{\sin x + x \cos x}
= \lim_{x \to 0^+} \frac{3 \cos x + [3(\cos x) + (3x + 1)(-\sin x)]}{\cos x + [3(\cos x) + (x)\sin x]}
= \lim_{x \to 0^+} \frac{6 \cos x - (3x + 1) \sin x}{6 \cos x - (3(0) + 1) \sin(0)} = \frac{6}{2} = 3 \square
\]
Theorem 4.5.B

Theorem 4.5.B. If \( \lim_{x \to a} \ln f(x) = L \) then
\[
\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{\lim_{x \to a} \ln f(x)} = e^L.
\]

Here, \( a \) may be finite or infinite.

Proof. Suppose \( \lim_{x \to a} \ln f(x) = L \). Then by the definition of limit, \( \ln f(x) \) is defined on some open interval \( I \) containing \( a \), except possibly at \( a \) itself. Since the natural logarithm is the inverse of the natural exponential, then \( e^{\ln f(x)} = f(x) \) on \( I \) except possibly at \( x = a \). Since the natural exponential function is continuous everywhere (in particular, at \( L \)) then
\[
\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^{\lim_{x \to a} \ln f(x)} = e^L,
\]
as claimed. \( \square \)

Exercise 4.5.52

Exercise 4.5.52. Use l'Hôpital's Rule (Theorem 4.6) to evaluate \( \lim_{x \to 1^+} x^{1/(x-1)} \). Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With \( f(x) = x \) and \( g(x) = 1/(x-1) \), we have \( \lim_{x \to 1^-} f(x) = 1 \) and \( \lim_{x \to 1^+} g(x) = \infty \), so \( \lim_{x \to 1^+} x^{1/(x-1)} \) is of the \( 1^\infty \) indeterminate form. We take a natural logarithm to get
\[
\lim_{x \to 1^+} \ln(x^{1/(x-1)}) = \lim_{x \to 1^+} \frac{1}{x-1} \ln x = \lim_{x \to 1^+} \frac{\ln x}{x-1}.
\]

\( \frac{0}{0} \)

\[
\lim_{x \to 1^+} \frac{\ln x}{x-1} = \frac{1/(1)}{1} = 1.
\]

So by Theorem 4.5.B, \( \lim_{x \to 1^+} x^{1/(x-1)} = e^{\lim_{x \to 1^+} \ln x^{1/(x-1)}} = e^1 = 2 \). \( \square \)

Exercise 4.5.58

Exercise 4.5.58. Use l'Hôpital's Rule (Theorem 4.6) to evaluate \( \lim_{x \to 0} (e^x + x)^{1/x} \). Write the indeterminate form over the equal sign when you use l'Hôpital's Rule.

Solution. With \( f(x) = e^x + x \) and \( g(x) = 1/x \), we have \( \lim_{x \to 0^-} f(x) = 1 \), \( \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} 1/x = \infty \), and \( \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} 1/x = -\infty \), so both \( \lim_{x \to 0^-} (e^x + x)^{1/x} \) and \( \lim_{x \to 0^+} (e^x + x)^{1/x} \) are of the \( 1^\infty \) indeterminate form. To evaluate \( \lim_{x \to 0^+} (e^x + x)^{1/x} \), we take a natural logarithm to get
\[
\lim_{x \to 0^+} \ln(e^x + x)^{1/x} = \lim_{x \to 0^+} (1/x) \ln(e^x + x) = \lim_{x \to 0^+} \frac{\ln(e^x + x)}{x}.
\]

\( \frac{0}{0} \)

\[
\lim_{x \to 0^+} \frac{\ln(e^x + x)}{x} = \lim_{x \to 0^+} \frac{1/(e^x + x) \cdot (e^x + 1)}{1} = \lim_{x \to 0^+} \frac{e^x + 1}{e^x + x} = \frac{e^0 + 1}{e^0 + 0} = 1.
\]

So by Theorem 4.5.B, \( \lim_{x \to 0} (e^x + x)^{1/x} = e^{\lim_{x \to 0^-} \ln(e^x + x)^{1/x}} = e^2 \). Therefore by Theorem 2.6, “Relation Between One-Sided and Two-Sided Limits,” \( \lim_{x \to 0} (e^x + x)^{1/x} = e^2 \). \( \square \)
**Exercise 4.5.81(b)**

**Exercise 4.5.81(b).** Use l'Hôpital’s Rule (Theorem 4.6) to evaluate

\[ \lim_{x \to \infty} (x - \sqrt{x^2 + x}) \]

Write the indeterminate form over the equal sign when you use l'Hôpital's Rule. HINT: As the first step, multiply by \((x + \sqrt{x^2 + x})/(x + \sqrt{x^2 + x})\) and simplify the new numerator.

**Solution.** Notice that \(\lim_{x \to \infty} x = \infty\) and \(\lim_{x \to \infty} \sqrt{x^2 + x} = \infty\), so that \(\lim_{x \to \infty} (x - \sqrt{x^2 + x})\) is of an \(\infty - \infty\) indeterminate form. We follow the hint and consider

\[
\lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right) = \lim_{x \to \infty} \left( \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \right) \left( \frac{x}{x} \right) = \lim_{x \to \infty} \frac{x}{x + \sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{-x}{x + \sqrt{x^2 + x}}
\]

**Theorem 4.7. Cauchy’s Mean Value Theorem.**

**Theorem 4.7. Cauchy’s Mean Value Theorem.**

Suppose functions \(f\) and \(g\) are continuous on \([a, b]\) and differentiable throughout \((a, b)\) and also suppose \(g'(x) \neq 0\) throughout \((a, b)\). Then there exists a number \(c\) in \((a, b)\) at which

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

**Proof.** First, notice that \(f\) and \(g\) both satisfy the hypotheses of the Mean Value Theorem (Theorem 4.4). We claim that \(g'(a) \neq g'(b)\), for if \(g(a) = g(b)\) then by the Mean Value Theorem we have

\[
g'(c) = \frac{g(b) - g(a)}{b - a} = 0 \text{ for some } c \in (a, b) \text{ contradicting the hypotheses of the theorem. Next, consider}
\]

\[
F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).
\]

Since \(f\) and \(g\) are continuous on \([a, b]\) then so is \(F\), since \(f\) and \(g\) are differentiable on \((a, b)\) then so is \(F\), and \(F(a) = F(b) = 0\).

**Solution (continued).**

\[
\lim_{x \to \infty} \left( x - \sqrt{x^2 + x} \right) = \lim_{x \to \infty} \frac{-x}{x + \sqrt{x^2 + x}}
\]

\[
\Rightarrow \lim_{x \to \infty} \frac{-1}{1 + (1/2)(x^2 + x)^{-1/2}[2x + 1]} = \lim_{x \to \infty} \frac{-1}{1 + (2x + 1)/(2\sqrt{x^2 + x})} = \frac{-1}{1 + 1} = \frac{-1}{2}
\]

because

\[
\lim_{x \to \infty} \frac{2x + 1}{2\sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{2x + 1}{2\sqrt{x^2 + x}} = \lim_{x \to \infty} \frac{(2x + 1)/x}{2\sqrt{x^2 + x}/x} = \lim_{x \to \infty} \frac{2 + 1/x}{2\sqrt{1 + 1/x}} = 2/2 = 1.
\]

**Theorem 4.7 (continued).**

**Theorem 4.7. Cauchy’s Mean Value Theorem.**

Suppose functions \(f\) and \(g\) are continuous on \([a, b]\) and differentiable throughout \((a, b)\) and also suppose \(g'(x) \neq 0\) throughout \((a, b)\). Then there exists a number \(c\) in \((a, b)\) at which

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

**Proof (continued).** Since \(f\) and \(g\) are continuous on \([a, b]\) then so is \(F\), since \(f\) and \(g\) are differentiable on \((a, b)\) then so is \(F\), and \(F(a) = F(b) = 0\). So by Rolle’s Theorem (Theorem 4.3) there is \(c \in (a, b)\) such that \(F'(c) = 0\). Since \(F'(x) = f'(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(x)\),

then \(F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0\) and hence

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}
\]

as claimed. \(\square\)
Suppose that \( f(a) = g(a) = 0 \), that \( f \) and \( g \) are differentiable on an open interval \( I \) containing \( a \), and that \( g'(x) \neq 0 \) on \( I \) if \( x \neq a \). Then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]
assuming that the limit on the right side of this equation exists.

Proof. We consider one-sided limits. Suppose \( x \to a^+ \) and \( x \in I \). Then \( g'(x) \neq 0 \), so by Cauchy's Mean Value Theorem (Theorem 4.7) applied on the interval \([a, x]\) we have for some \( c \in (a, x) \) that
\[
\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.
\]
Since \( f(a) = g(a) = 0 \) by hypothesis, then \( \frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}. \) Notice that as \( x \to a^+ \) then \( c \to a^+ \) (since for any given \( x \), the corresponding \( c \) is between \( a \) and \( x \)).

Theorem 4.6 (continued).
Suppose that \( f(a) = g(a) = 0 \), that \( f \) and \( g \) are differentiable on an open interval \( I \) containing \( a \), and that \( g'(x) \neq 0 \) on \( I \) if \( x \neq a \). Then
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},
\]
assuming that the limit on the right side of this equation exists.

Proof (continued). Therefore
\[
\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(c)}{g'(c)},
\]
so l'Hôpital's Rule holds as \( x \to a^+ \). The same argument (except with Cauchy's Mean Value Theorem applied on the interval \([x, a]\)) shows that l'Hôpital's Rule holds as \( x \to a^- \) also. So by Theorem 2.6, "Relation Between One-Sided and Two-Sided Limits," the claim holds. \( \square \)