Exercise 4.6.2 (continued)

Exercise 4.6.2. Show that among all rectangles with an 8-m perimeter, the one with the largest area is a square.

Solution. What is the maximum of $A(x) = x(4 - x) = 4x - x^2$ for $x \in [0, 4]$? We have $A'(x) = 4 - 2x$, so the critical point is $x = 2$. As in Section 4.1 (using the technique based on Theorem 4.2, “Local Extreme Values”), we test the endpoints and critical points:

- $A(0) = 4(0) - (0)^2 = 0$, $A(2) = 4(2) - (2)^2 = 4$, and $A(4) = 4(4) - (4)^2 = 0$. So the maximum area is 4 m$^2$ and occurs when $x = 2$ m and $y = 4 - x = 4 - (2) = 2$ m. Since the largest area rectangle has width $x = 2$ m and the height $y = 2$ m, then the largest area rectangle is a square, as claimed. □

Exercise 4.6.8

Exercise 4.6.8. The Shortest Fence.

A 216 m$^2$ rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Solution. (2 and 3) We consider a rectangle with width $x$ and height $y$. We divide the field in half as follows:

Since the area of the field is 216 m$^2$, then we have $xy = 216$ m$^2$, or $y = 216/x$. 
Solution (continued). (4) The amount of fencing in terms of \( x \) and \( y \) is
\[
F = 2x + 3y.
\]
So as a function of \( x \) we have
\[
F(x) = 2x + 3(216/x) = 2x + 648/x = 2x + 648x^{-1}.
\]
(5) The question is: What is the minimum of \( F \) for \( x \in (0, \infty) \). We have
\[
F'(x) = 2 - 648/x^2 = (2x^2 - 648)/x^2 = 2(x^2 - 324)/x^2.
\]
So the critical points are \( x = \pm \sqrt{324} = \pm 18 \), but only the critical point \( x = 18 \) is in
the interval \((0, \infty)\). We apply the First Derivative Test for Local Extrema (Theorem 4.3.A) and test the sign of \( F' \) as in Section 4.3:

<table>
<thead>
<tr>
<th>interval</th>
<th>(0, 18)</th>
<th>(18, \infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value ( k )</td>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>( F(k) )</td>
<td>2((1^2 - 324)/(1)^2) = -646</td>
<td>2((20)^2 - 324)/(20)^2 = 76/400</td>
</tr>
<tr>
<td>( F'(x) )</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( F(x) )</td>
<td>DEC</td>
<td>INC</td>
</tr>
</tbody>
</table>

So \( F \) has a local minimum at \( x = 18 \).

Exercise 4.6.12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

Solution. We follow the steps.

(2 and 3) We use the picture in the book. With the book’s variables, the height of the cone is \( h = y + 3 \) and the radius is \( r = x \). From the Pythagorean Theorem, \( x^2 + y^2 = 3^2 \) or \( x^2 = 9 - y^2 \). Since \( x \) and \( y \) are distances (and hence non-negative), then we must have \( y \in [0, 3] \) (one might argue that \( y \in [-3, 3] \), but a maximum volume clearly does not happen for \( y \in [-3, 0) \)).

(4) The volume of a cone is \( V = \frac{1}{3} \pi r^2 h \), so \( V = \frac{1}{3} \pi x^2 (y + 3) \), or
\[
V(y) = \frac{1}{3} \pi (9 - y^2)(y + 3) = \frac{1}{3} \pi (-y^3 - 3y^2 + 9y + 27).
\]

Solution (continued). (5) The question is: Find the maximum of
\[
V(y) = \frac{1}{3} \pi (-y^3 - 3y^2 + 9y + 27) \text{ for } y \in [0, 3].
\]
We have
\[
V'(y) = \frac{1}{3} \pi (-3y^2 - 6y + 9) = \pi (-y^2 - 2y + 3) = \pi (-y + 1)(y + 3),
\]
and the critical points of \( V \) are \( y = 1 \) and \( y = -3 \). So as in Section 4.1, we consider the endpoints and critical points in \([0, 3] \). We have
\[
V(0) = \frac{1}{3} \pi (-0)^3 - 3(0)^2 + 9(0) + 27 = 9 \pi,
\]
\[
V(1) = \frac{1}{3} \pi (-1)^3 - 3(1)^2 + 9(1) + 27 = \frac{32}{3} \pi,
\]
and
\[
V(3) = \frac{1}{3} \pi (-3)^3 - 3(3)^2 + 9(3) + 27 = 0.
\]
So the maximum volume is \( 32 \pi/3 \) (cubic units). Notice that the occurs when the \( y = 1 \) and \( x = \sqrt{9 - y^2} = \sqrt{9 - (1)^2} = \sqrt{8} = 2 \sqrt{2} \).
Exercise 4.6.24

Exercise 4.6.24. The trough in the figure is to be made to the dimensions shown. Only the angle $\theta$ can be varied. What value of $\theta$ will maximize the trough’s volume?

Solution. (2 and 3) We start with the picture in the book. With the book’s variable, we have:

Since $\theta$ is in a right triangle then $\theta \in [0, \pi/2]$.

Exercise 4.6.24 (continued 1)

Solution. (4) We need the volume of the trough. We can think of the cross-section of the trough as two triangles, each of base $\sin \theta$ and height $\cos \theta$, and a rectangle of base 1 and height $\cos \theta$. So the cross-sectional area of the trough is $2 \left(\frac{1}{2} \sin \theta \cos \theta \right) + (1)(\cos \theta) = \sin \theta \cos \theta + \cos \theta \, \text{ft}^2$. Hence the volume of the trough is $V(\theta) = 20(\sin \theta \cos \theta + \cos \theta) \, \text{ft}^3$.

(5) The question is: Find the maximum of $V(\theta) = 20(\sin \theta \cos \theta + \cos \theta)$ for $\theta \in [0, \pi/2]$. We have

$$V'(\theta) = 20[(\cos \theta)(\cos \theta) + (\sin \theta)(-\sin \theta) + (-\sin \theta)] = 20(\cos^2 \theta - \sin^2 \theta - \sin \theta) = 20(1 - \sin^2 \theta - \sin \theta) = 20(1 - \sin \theta - 2\sin^2 \theta) = 20(1 + \sin \theta)(1 - 2\sin \theta).$$

Exercise 4.6.24 (continued 2)

Solution. (5) We have $V'(\theta) = 20(1 + \sin \theta)(1 - 2\sin \theta)$. So the relevant critical points of $V$ are $\theta \in [0, \pi/2]$ for which $\sin \theta = -1$ or $\sin \theta = 1/2$. There are no $\theta \in [0, \pi/2]$ such that $\sin \theta = -1$. For $\sin \theta = 1/2$ and $\theta \in [0, \pi/2]$ we have $\theta = \pi/6$, so that the only critical point in $[0, \pi/2]$ is $\pi/6$. So as in Section 4.1, we consider the endpoints and critical points in $[0, \pi/2]$. We have $V(0) = 20(\sin(0)\cos(0) + \cos(0)) = 20$, $V(\pi/6) = 20(\sin(\pi/6)\cos(\pi/6) + \cos(\pi/6)) = 20((1/2)(\sqrt{3}/2) + \sqrt{3}/2) = 20(3\sqrt{3}/4 + \sqrt{3}/2) = 5\sqrt{3} + 10\sqrt{3} \approx 29.98$, and $V(\pi/2) = 20(\sin(\pi/2)\cos(\pi/2) + \cos(\pi/2)) = 0$. So the maximum of $V$ occurs when $\theta = \pi/6$ and the maximum volume is $5\sqrt{3} + 10\sqrt{3} \, \text{ft}^3$. □

Exercise 4.6.42

Exercise 4.6.42. A cone is formed from a circular piece of material of radius 1 meter by removing a section of angle $\theta$ and then joining the two straight edges. Determine the largest possible volume for the cone.

Solution. (2 and 3) We use the picture in the book, but we introduce height $h$ and radius $r$ of the cone as variables. Notice that $h^2 + r^2 = 1^2$ or $h = \sqrt{1 - r^2}$. The circumference of the base of the cone is $2\pi r$, and this equals the length of the arc subtended by the angle $2\pi - \theta$ in the circle of radius 1 (namely, the length $(2\pi - \theta)(1) = 2\pi - \theta$); notice that $\theta \in [0, 2\pi]$. So we have $2\pi r = 2\pi - \theta$ or $r = 1 - \theta/(2\pi)$. 

Solution. (4) We need the volume of the cone. We can think of the cross-section of the cone as two circles, each of radius $\sin \theta$ and height $\cos \theta$, and a rectangle of base 1 and height $\cos \theta$. So the cross-sectional area of the cone is $2 \left(\frac{1}{2} \sin \theta \cos \theta \right) + (1)(\cos \theta) = \sin \theta \cos \theta + \cos \theta \, \text{ft}^2$. Hence the volume of the cone is $V(\theta) = 20(\sin \theta \cos \theta + \cos \theta) \, \text{ft}^3$.
Exercise 4.6.42 (continued 1)

Solution (continued). Notice that this gives \( h = \sqrt{1 - r^2} = \sqrt{1 - (1 - \theta/(2\pi))^2} = \sqrt{1 - (1 - \theta/\pi + \theta^2/(4\pi^2))} = \sqrt{\theta/\pi - \theta^2/(4\pi^2)}. \)

(4) The volume of the cone is \( V = \frac{1}{3} \pi r^2 h. \) In terms of \( \theta, \)

\[
V(\theta) = \frac{1}{3} \pi (1 - \theta/(2\pi))^2 \sqrt{\theta/\pi - \theta^2/(4\pi^2)} = \frac{\pi}{3} \left( \frac{2\pi - \theta}{2\pi} \right)^2 \sqrt{\frac{4\pi \theta - \theta^2}{4\pi^2}}
\]

\[
= \frac{\pi}{24\pi^2} (2\pi - \theta)^2 \sqrt{4\pi \theta - \theta^2} = \frac{1}{24\pi^2} (2\pi - \theta)^2 (4\pi \theta - \theta^2)^{1/2}.
\]

(5) The question is: Find the maximum of \( V(\theta) = \frac{1}{24\pi^2} (2\pi - \theta)^2 (4\pi \theta - \theta^2)^{1/2} \) for \( \theta \in [0, 2\pi]. \) We have

\[
V'(\theta) = \frac{1}{24\pi^2} \left( [2(2\pi - \theta)\sqrt{4\pi \theta - \theta^2}] + (2\pi - \theta)^2 \left[ \frac{1}{2} (4\pi \theta - \theta^2)^{-1/2} [4\pi - 2\theta] \right] \right)
\]

Exercise 4.6.42 (continued 2)

Solution (continued).

\[
= \frac{1}{24\pi^2} (2\pi - \theta) \left( 2\sqrt{4\pi \theta - \theta^2} + \frac{(2\pi - \theta)^2}{\sqrt{4\pi \theta - \theta^2}} \right)
\]

\[
= \frac{1}{24\pi^2} (2\pi - \theta) \left( -2(4\pi \theta - \theta^2) + \frac{(2\pi - \theta)^2}{\sqrt{4\pi \theta - \theta^2}} \right)
\]

\[
= \frac{2\pi - \theta}{24\pi^2 \sqrt{4\pi \theta - \theta^2}} \left( -8\pi \theta + 2\theta^2 + 4\pi^2 - 4\pi \theta + \theta^2 \right)
\]

\[
= \frac{(2\pi - \theta)(3\theta^2 - 12\pi \theta + 4\pi^2)}{24\pi^2 \sqrt{4\pi \theta - \theta^2}} = V'(\theta).
\]

So we look for critical points in \([0, 2\pi]\) and notice first that the endpoints \( \theta = 0 \) and \( \theta = 2\pi \) are both critical points. Notice that \( 4\pi \theta - \theta^2 \) is nonnegative for \( \theta \in [0, 2\pi] \) so no critical points result from the term \( \sqrt{4\pi \theta - \theta^2}. \)

Exercise 4.6.42 (continued 3)

Solution (continued). ...

\[
V'(\theta) = \frac{(2\pi - \theta)(3\theta^2 - 12\pi \theta + 4\pi^2)}{24\pi^2 \sqrt{4\pi \theta - \theta^2}}.
\]

So we next set \( V'(\theta) = 0 \) and set the factor \( (3\theta^2 - 12\pi \theta + 4\pi^2) \) equal to 0. This implies from the quadratic formula that

\[
\theta = \frac{-(-12\pi) \pm \sqrt{(-12\pi)^2 - 4(3)(4\pi^2)}}{2(3)} = \frac{12\pi \pm \sqrt{144\pi^2 - 48\pi^2}}{6}
\]

\[
= \frac{12\pi \pm \sqrt{96\pi^2}}{6} = \frac{12\pi \pm 4\sqrt{6}\pi}{6} = 2\pi \pm \frac{2\sqrt{6}\pi}{3}.
\]

So the only critical point in \((0, 2\pi)\) is \( \theta = 2\pi - 2\sqrt{6}\pi/3. \)

Exercise 4.6.42 (continued 4)

Solution (continued). Since \( V(\theta) = \frac{1}{24\pi^2} (2\pi - \theta)^2 (4\pi \theta - \theta^2)^{1/2}, \) then we have \( V(0) = 0, V(2\pi) = 0, \) and \( V \left( 2\pi - \frac{2\sqrt{6}\pi}{3} \right) = \)

\[
\frac{1}{24\pi^2} \left( 2\pi - \left( 2\pi - \frac{2\sqrt{6}\pi}{3} \right) \right)^2 \left( 4\pi \left( 2\pi - \frac{2\sqrt{6}\pi}{3} \right) - \left( 2\pi - \frac{2\sqrt{6}\pi}{3} \right)^2 \right)^{1/2}
\]

\[
= \frac{1}{24\pi^2} \left( \frac{2\sqrt{6}\pi}{3} \right)^2 \left( 8\pi^2 - 8\sqrt{6} \pi^2 - 4\pi^2 + \frac{8\sqrt{6}\pi^2}{3} - 24\pi^2 \right)^{1/2}
\]

\[
= \frac{1}{9} (4 - 24/9)^{1/2} \pi = \frac{1}{9} \sqrt{\frac{36 - 24}{9}} \pi = \sqrt{\frac{12}{27}} \pi = \frac{2\sqrt{3}}{27} \pi.
\]

So \( V \) is a maximum of \( \frac{2\sqrt{3}}{27} \pi \) when \( \theta = 2\pi - 2\sqrt{6}\pi/3. \) □
Exercise 4.6.52

Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively. (a) At what times in the interval $t > 0$ do the masses pass each other? HINT: $2 \sin t = 2 \sin t \cos t$.

(b) When in the interval $0 \leq t \leq 2\pi$ is the vertical distance between the masses the greatest? What is the distance? HINT: $\cos 2t = 2 \cos^2 t - 1$.

Solution. (a) The masses pass each other when $s_1 = s_2$, so we consider the equation $2 \sin t = 2 \sin t \cos t$ or (by the hint) $2 \sin t = 2 \sin t \cos t$ or $2 \sin t - 2 \sin t \cos t = 0$ or $2 \sin t (1 - \cos t) = 0$. So we need either $\sin t = 0$ or $\cos t = 1$. We have $\sin t = 0$ for $t = n\pi$ where $n \in \mathbb{Z}$. We have $\cos t = 1$ for $t = 2n\pi$ where $n \in \mathbb{Z}$. So the masses pass each other for $t \in \{n\pi | n \in \mathbb{Z}\}$, which for $t > 0$ gives $t \in \{n\pi | n \in \mathbb{N}\}$. □

Solution (continued). ... $D_2'(t) = 4(2 \sin t - \sin 2t)(\cos t - \cos 2t)$. We know that $2 \sin t - \sin 2t = 0$ for $t \in \{0, \pi, 2\pi\}$ (since $t \in [0, 2\pi]$). We have by the hint that $\cos t - \cos 2t = \cos t - (2 \cos^2 t - 1) = -2 \cos^2 t + \cos t + 1 = (-2 \cos t - 1)(\cos t - 1)$, so we also have critical points when $\cos t = 1$ or $\cos t = -1$. We have $-2 \cos t = 1 = 0$, or $\cos t = -1/2$ which implies $t = 2\pi/3$ or $t = 4\pi/3$. We have $\cos t = 1$ for $t = 0$ or $t = 2\pi$. So as in Section 4.1, we consider the endpoints and critical points in $[0, 2\pi]$. We have $D_2(0) = D_2(\pi) = D_2(2\pi) = (2 \sin(0) - \sin(2(0)))^2 = (0)^2 = 0$, $D_2(2\pi/3) = (2 \sin(2\pi/3) - \sin(2(2\pi/3)))^2 = (2(\sqrt{3}/2) - (-\sqrt{3}/2))^2 = (3\sqrt{3}/2)^2$, and. $D_2(4\pi/3) = (2 \sin(4\pi/3) - \sin(2(4\pi/3)))^2 = (2(-\sqrt{3}/2) - (\sqrt{3}/2))^2 = (-3\sqrt{3}/2)^2$ So $D_2$ has a maximum at $\theta = 2\pi/3$ and at $\theta = 4\pi/3$. Hence the distance is greatest when $\theta = 2\pi/3$ or $\theta = 4\pi/3$ and that distance is $\sqrt{D_2(2\pi/3)} = \sqrt{D_2(4\pi/3)} = 3\sqrt{3}/2$. □

Chapter 4 Practice Exercise 112

The Ladder Problem.

What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.

HINT: Write the length of a ladder that is wedged into the hallway as $L = L_1 + L_2$ and introduce an angle $\theta$ as shown here. Then find the minimum value of $L$ to determine the longest ladder that will pass around the corner.
Chapter 4 Practice Exercise 112

Solution. (2 and 3) We use the picture above. Notice that $\theta \in (0, \pi/2)$. We have $L = L_1 + L_2$, $\sin \theta = 6/L_1$, and $\cos \theta = 8/L_2$. So $L_1 = 6/\sin \theta$ and $L_2 = 8/\cos \theta$.

(4) We have $L = L_1 + L_2$, so that from (3)
$L(\theta) = 6/\sin \theta + 8/\cos \theta = 6 \csc \theta + 8 \sec \theta$.

(5) The question is: Find the minimum of $L(\theta)$ for $\theta \in (0, \pi/2)$.

Chapter 4 Practice Exercise 112 (continued 2)

Solution (continued). We have $L'(\theta) = -6 \cos^3 \theta + 8 \sin^3 \theta$ (and the denominator is positive for all $\theta \in (0, \pi/2)$). With $\theta_k = \tan^{-1}(\sqrt{3}/8)$ we have $\tan \theta_k = \sqrt{3}/8$ or $\tan^3 \theta_k = 3/8$ or $\sin \theta_k / \cos \theta_k = 3/8$ or $\sin \theta_k = (3/8) \cos \theta_k$, and hence

$-6 \cos^3 \theta_k + 8 \sin^3 \theta_k = -6 \cos^3 \theta_k + 8((3/8) \cos \theta_k)^3 = (-357/64) \cos^3 \theta_k < 0$.

So $L'(\tan^{-1}(\sqrt{3}/8)) < 0$. With $\theta_k = \pi/4$ we have

$-6 \cos^3 \theta_k + 8 \sin^3 \theta_k = -6 \cos^3(\pi/4) + 8 \sin^3(\pi/4) = -6(\sqrt{2}/2)^3 + 8(\sqrt{2}/2)^3 = 2(\sqrt{2}/2)^3 > 0$. So $L'(\pi/4) > 0$. We now test the sign of $L'$ as follows:

<table>
<thead>
<tr>
<th>interval</th>
<th>$0, \tan^{-1}(\sqrt{3}/4)$</th>
<th>$\tan^{-1}(\sqrt{3}/4), \pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value $\theta_k$</td>
<td>$\tan^{-1}(\sqrt{3}/8)$</td>
<td>$\pi/4$</td>
</tr>
<tr>
<td>$L'(\theta)$</td>
<td>DEC</td>
<td>INC</td>
</tr>
<tr>
<td>$L(\theta)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Chapter 4 Practice Exercise 112 (continued 3)

Solution (continued). So by the First Derivative Test for Local Extrema (Theorem 4.3A), $L$ has a local minimum at $\theta = \tan^{-1}(\sqrt{3}/4)$.

Since $1 + \tan^2 \theta = \sec^2 \theta$ then $1 + (\sqrt{3}/4)^2 = \sec^2 \theta$ or

$\cos^2 \theta = \frac{1}{1 + 9/16} = \frac{\sqrt{16}}{\sqrt{16} + \sqrt{9}} = \frac{2\sqrt{2}}{2\sqrt{2} + \sqrt{9}}$.

or $\cos \theta = \sqrt{\frac{2\sqrt{2}}{2\sqrt{2} + \sqrt{9}}}$ (notice that $\theta$ is acute so $\cos \theta > 0$).
Solution (continued). Since \( \sin^2 \theta = 1 - \cos^2 \theta \) then

\[
\sin^2 \theta = 1 - \left( \frac{2\sqrt{2}}{2\sqrt{2} + \sqrt{9}} \right)^2 = 1 - \frac{2\sqrt{2}}{2\sqrt{2} + \sqrt{9}} \\
= \frac{(2\sqrt{2} + \sqrt{9}) - 2\sqrt{2}}{2\sqrt{2} + \sqrt{9}} = \frac{\sqrt{9}}{2\sqrt{2} + \sqrt{9}} \quad \text{or} \quad \sin \theta = \sqrt{\frac{\sqrt{9}}{2\sqrt{2} + \sqrt{9}}}.
\]

So at the critical point \( \theta = \tan^{-1}(\sqrt{3}/4) \),

\[
L(\theta) = \frac{6}{\sin \theta} + \frac{8}{\cos \theta} = 6\sqrt{2\sqrt{2} + \sqrt{9}} + 8\sqrt{2\sqrt{2} + \sqrt{9}} \\
= \sqrt{2\sqrt{2}/9 + 1} + 8\sqrt{1 + (1/2)\sqrt{9}/2} \approx 19.7313 \text{ ft.}
\]

Rounding down to the nearest foot, the ladder can be up to 19 ft long. □

Exercise 4.6.62

Suppose that \( c(x) = x^3 - 20x^2 + 20,000x \) is the cost of manufacturing \( x \) items. Find a production level that will minimize the average cost \( c(x)/x \) of making \( x \) items.

Solution. (2 and 3) There is no picture and all variables are stated in the question.

(4) The average cost function is \( A(x) = c(x)/x = (x^3 - 20x^2 + 20,000x)/x = x^2 - 20x + 20,000 \).

(5) The question is: Minimize \( A(x) = c(x)/x \) for \( x \in (0, \infty) \). We have \( A'(x) = 2x - 20 \), so that \( x = 10 \) is the only critical point. Notice that \( A''(x) = 2 \) and, in particular, \( A''(10) = 2 > 0 \) so that by the Second Derivative Test for Local Extrema (Theorem 4.5) \( A \) has a local minimum at \( x = 10 \). Since \( A \) has no other critical points, then \( x = 10 \) is an absolute minimum. A production level of [10 items] will minimize average cost. □