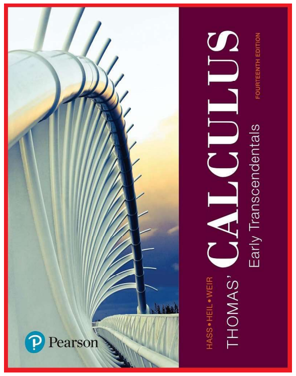
Calculus 1

Chapter 5. Integrals

5.2. Sigma Notation and Limits of Finite Sums—Examples and Proofs



Exercise 5.2.2

Exercise 5.2.2. Write the sum $\sum_{k=1}^{3} \frac{k-1}{k}$ without the sigma notation and then evaluate the sum.

Solution. We have

$$\sum_{k=1}^{3} \frac{k-1}{k} = \frac{(1)-1}{(1)} + \frac{(2)-1}{(2)} + \frac{(3)-1}{(3)} = 0 + \frac{1}{2} + \frac{2}{3} = \boxed{\frac{7}{6}}. \ \square$$

Exercise 5.2.12

Exercise 5.2.12. Express the sum $1 + 4 + 9 + 16$ in sigma notation.

Solution. Notice that these numbers 1, 4, 9, and 16 are the squares of the natural numbers 1, 2, 3, and 4 (respectively). So we have:

$$1 + 4 + 9 + 16 = 1^2 + 2^2 + 3^2 + 4^2 = \boxed{\sum_{k=1}^{4} k^2}. \ \square$$

Exercise 5.2.18

Exercise 5.2.18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of: **(a)** $\sum_{k=1}^{n} 8a_k$, **(b)** $\sum_{k=1}^{n} 250b_k$, **(c)** $\sum_{k=1}^{n} (a_k + 1)$, and **(d)** $\sum_{k=1}^{n} (b_k - 1)$.

Solution. (a) We have

$$\begin{aligned} \sum_{k=1}^{n} 8a_k &= 8 \sum_{k=1}^{n} a_k \text{ by Theorem 5.2.A(3), “Constant Multiple Rule”} \\ &= 8(0) = \boxed{0} \text{ since } \sum_{k=1}^{n} a_k = 0. \end{aligned}$$

Exercise 5.2.18 (continued 1)

Exercise 5.2.18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of: **(b)** $\sum_{k=1}^{n} 250b_k$, **(c)** $\sum_{k=1}^{n}(a_k+1)$, and **(d)** $\sum_{k=1}^{n}(b_k-1)$.

Solution. **(b)** We have

$$\begin{aligned}\sum_{k=1}^{n} 250b_k &= 250\sum_{k=1}^{n} b_k \text{ by Theorem 5.2.A(3), ``Constant Multiple Rule''}\\ &= 250(1) = \boxed{250} \text{ since } \sum_{k=1}^{n} b_k = 1.\end{aligned}$$

Exercise 5.2.18 (continued 2)

Exercise 5.2.18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of: **(c)** $\sum_{k=1}^{n}(a_k+1)$, and **(d)** $\sum_{k=1}^{n}(b_k-1)$.

Solution. **(c)** We have

$$\begin{aligned}\sum_{k=1}^{n}(a_k+1) &= \sum_{k=1}^{n}(a_k) + \sum_{k=1}^{n}(1) \text{ by Theorem 5.2.A(1), ``Sum Rule''}\\ &= (0) + n(1) \text{ since } \sum_{k=1}^{n} a_k = 0 \text{ and } \sum_{k=1}^{n}(1) = n(1) = n\\ &\quad\text{by Theorem 5.2.A(4), ``Constant Value Rule''}\\ &= \boxed{n}.\end{aligned}$$

Exercise 5.2.18 (continued 3)

Exercise 5.2.18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of: **(d)** $\sum_{k=1}^{n}(b_k-1)$.

Solution. **(d)** We have

$$\begin{aligned}\sum_{k=1}^{n}(b_k-1) &= \sum_{k=1}^{n}(b_k) + \sum_{k=1}^{n}(-1) \text{ by Theorem 5.2.A(1), ``Sum Rule''}\\ &= (1) + n(-1) \text{ since } \sum_{k=1}^{n} b_k = 1 \text{ \& } \sum_{k=1}^{n}(-1) = n(-1) = -n\\ &\quad\text{by Theorem 5.2.A(4), ``Constant Value Rule''}\\ &= \boxed{1-n}. \ \square\end{aligned}$$

Exercise 5.2.24

Exercise 5.2.24. Evaluate the sum using Theorem 5.2.B: $\sum_{k=1}^{6}(k^2-5)$.

Solution. We have

$$\begin{aligned}\sum_{k=1}^{6}(k^2-5) &= \sum_{k=1}^{6} k^2 - \sum_{k=1}^{6} 5 \text{ by Theorem 5.2.A(2), ``Difference Rule''}\\ &= \frac{(6)((6)+1)(2(6)+1)}{6} - 6(5) \text{ since}\\ &\quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \text{ by Theorem 5.2.B(2) and}\\ &\quad \sum_{k=1}^{6}(5) = 6(5) = 30 \text{ by Thm 5.2.A(4), Const. Mult. Rule}\\ &= \frac{(6)(7)(13)}{6} - 30 = 91 - 30 = \boxed{61}. \ \square\end{aligned}$$

Exercise 5.2.28

Exercise 5.2.28. Evaluate the sum using Theorem 5.2.B:

$$\left(\sum_{k=1}^{7} k\right) - \sum_{k=1}^{7} \frac{k^3}{4}.$$

Solution. We have

$$\left(\sum_{k=1}^{7} k\right) - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right) - \frac{1}{4}\sum_{k=1}^{7} k^3$$

by Theorem 5.2.A(3), "Constant Multiple Rule"

Exercise 5.2.28 (continued)

Solution. We have

$$\left(\sum_{k=1}^{7} k\right) - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right) - \frac{1}{4}\sum_{k=1}^{7} k^3$$

by Theorem 5.2.A(3), "Constant Multiple Rule"

$$= \frac{(7)((7)+1)}{2} - \frac{1}{4}\left(\frac{(7)((7)+1)}{2}\right)^2$$

since $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ by Theorem 5.2.B(1)

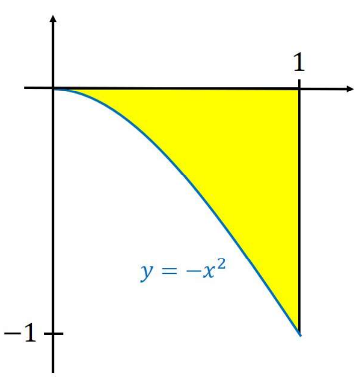
and $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$ by Theorem 5.2.B(3)

$$= 28 - 196 = \boxed{-168}. \quad \square$$

Exercise 5.2.38

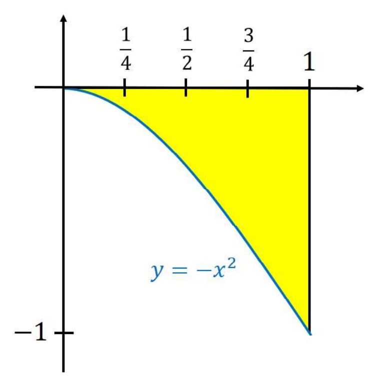
Exercise 5.2.38. Graph function $f(x) = -x^2$ over interval $[0, 1]$. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\sum_{k=1}^{4} f(c_k)\Delta x$, given that c_k is the **(a)** left-hand endpoint, **(b)** right-hand endpoint, **(c)** midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

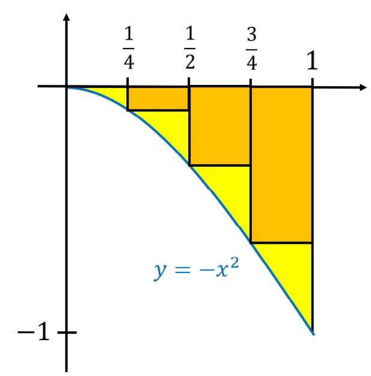
Solution. The graph of $f(x) = -x^2$ over interval $[0, 1]$, along with the "area" between the curve and the x-axis, are:



Exercise 5.2.38 (continued 1)

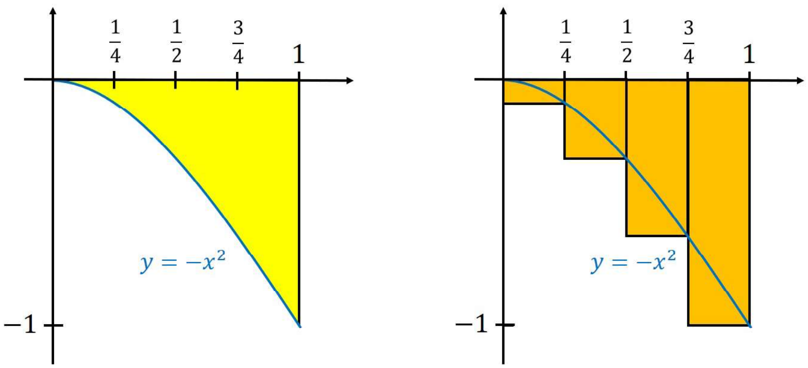
Solution (continued). (a) The graph and the partitioning of the interval is given here (left), along with the rectangles based on left-endpoints (right):





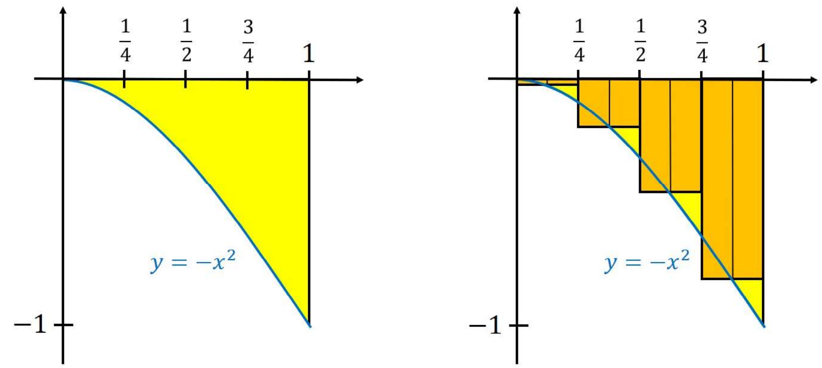
Exercise 5.2.38 (continued 2)

Solution (continued). (b) The graph and the partitioning of the interval is given here (left), along with the rectangles based on right-endpoints (right):



Exercise 5.2.38 (continued 3)

Solution (continued). (c) The graph and the partitioning of the interval is given here (left), along with the rectangles based on midpoints (right):



□

Example 5.2.5

Example 5.2.5. Partition the interval $[0, 1]$ into n subintervals of the same width, give the lower sum approximation of area under $y = 1 - x^2$ based on n, and find the limit as $n \to \infty$ (in which case the width of the subintervals approaches 0).

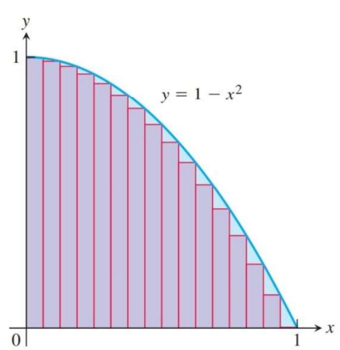
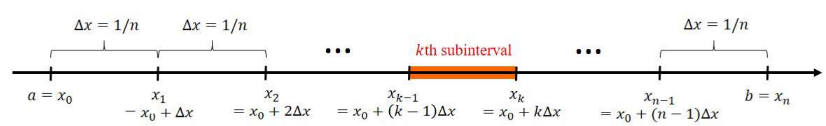


Figure 5.4(a)

Example 5.2.5 (continued 1)

Example 5.2.5. Partition the interval $[0, 1]$ into n subintervals of the same width, give the lower sum approximation of area under $y = 1 - x^2$ based on n, and find the limit as $n \to \infty$ (in which case the width of the subintervals approaches 0).

Solution. If we partition the interval $[a, b] = [0, 1]$ into n subintervals of the same width, then that width will be $\Delta x = (b - a)/n = (1 - 0)/n = 1/n$. The resulting subintervals will be $[x_{k-1}, x_k]$ for $k = 1, 2, \ldots, n$, where $x_k = a + k\Delta x = 0 + k(1/n) = k/n$ for $k = 0, 1, \ldots, n$.



Since $y = f(x) = 1 - x^2$ is a decreasing function, we use the right-hand endpoint in determining the function value used for a given subinterval. That is, we take $c_k = x_k = k/n$.

Example 5.2.5 (continued 2)

Example 5.2.5. Partition the interval $[0,1]$ into n subintervals of the same width, give the lower sum approximation of area under $y = 1 - x^2$ based on n, and find the limit as $n \to \infty$ (in which case the width of the subintervals approaches 0).

Solution (continued). With $c_k = x_k = k/n$ and $\Delta x_k = \Delta x = 1/n$ (when Δx_k is the same for all k, the partition is called *regular*), we have the Riemann sum:

$$s_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(k/n)\,(1/n) = \sum_{k=1}^{n}(1-(k/n)^2)\,(1/n)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k^2}{n^2}\right) = \frac{1}{n}\sum_{k=1}^{n}(1) - \frac{1}{n^3}\sum_{k=1}^{n}k^2$$

$$= \frac{1}{n}(n) - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \text{ by Theorem 5.2.B(2)}$$

Example 5.2.5 (continued 3)

Solution (continued). ...

$$s_n = \frac{1}{n}(n) - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = 1 - \frac{(n+1)(2n+1)}{6n^2}.$$

The limit as $n \to \infty$ of the Riemann sum is:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{(n+1)(2n+1)}{6n^2}\right) = 1 - \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2}$$

$$= 1 - \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2}\left(\frac{1/n^2}{1/n^2}\right) = 1 - \lim_{n\to\infty}\frac{(2n^2+3n+1)/n^2}{6n^2/n^2}$$

$$= 1 - \lim_{n\to\infty}\frac{2+3/n+1/n^2}{6} = 1 - \frac{2+3\lim_{n\to\infty}(1/n)+(\lim_{n\to\infty}1/n)^2}{6}$$

$$= 1 - \frac{2+3(0)+(0)^2}{6} = 1 - \frac{2}{6} = \boxed{\frac{2}{3}}. \ \square$$

Exercise 5.2.48

Exercise 5.2.48. For the function $f(x) = 3x + 2x^2$, find a formula for the Riemann sum obtained by dividing the interval $[a,b] = [0,1]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[0,1]$.

Solution. If we partition the interval $[a,b] = [0,1]$ into n subintervals of the same width, then that width will be $\Delta x = (b-a)/n = (1-0)/n = 1/n$. The resulting subintervals will be $[x_{k-1}, x_k]$ for $k = 1,2,\ldots,n$, where $x_k = a + k\Delta x = 0 + k(1/n) = k/n$ for $k = 0,1,\ldots,n$. Using the right-hand endpoint for c_k, we have $c_k = x_k = k/n$. We have the Riemann sum:

$$s_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(k/n)\,(1/n) = \sum_{k=1}^{n}(3(k/n)+2(k/n)^2)\,(1/n)$$

Exercise 5.2.48 (continued 1)

Solution (continued).

$$s_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(k/n)\,(1/n) = \sum_{k=1}^{n}(3(k/n)+2(k/n)^2)\,(1/n)$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{3}{n}k + \frac{2}{n^2}k^2\right) \text{ by Theorem 5.2.A(3)}$$

$$= \frac{3}{n^2}\sum_{k=1}^{n}k + \frac{2}{n^3}\sum_{k=1}^{n}k^2 \text{ by Theorem 5.2.A(1,3)}$$

$$= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \text{ by Theorem 5.2.B(1,2)}$$

$$= \frac{3(n+1)}{2n} + \frac{(n+1)(2n+1)}{3n^2}.$$

Exercise 5.2.48 (continued 2)

Solution (continued). Taking a limit as $n \to \infty$ of the Riemann sum gives:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{3(n+1)}{2n} + \frac{(n+1)(2n+1)}{3n^2}\right)$$

$$= \lim_{n\to\infty}\frac{3(n+1)}{2n} + \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2}$$

$$= \lim_{n\to\infty}\frac{3(n+1)}{2n}\left(\frac{1/n}{1/n}\right) + \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2}\left(\frac{1/n^2}{1/n^2}\right)$$

$$= \lim_{n\to\infty}\frac{3(n+1)/n}{2n/n} + \lim_{n\to\infty}\frac{2n^2/n^2 + 3n/n^2 + 1/n^2}{3n^2/n^2}$$

$$= \lim_{n\to\infty}\frac{3+1/n}{2} + \lim_{n\to\infty}\frac{2+3/n+1/n^2}{3} = \frac{3+(0)}{2} + \frac{2+3(0)+(0)^2}{3}$$

$$= \frac{3}{2} + \frac{2}{3} = \boxed{\frac{13}{6}}. \quad \square$$