Theorem 5.3

Theorem 5.3. The Mean Value Theorem for Definite Integrals. If \( f \) is continuous on \([a, b]\), then at some point \( c \) in \([a, b]\),

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

Proof. By the Max-Min Inequality (Theorem 5.2(6)), we have

\[
\min f \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \max f.
\]

Since \( f \) is continuous, \( f \) must assume any value between \( \min f \) and \( \max f \), including \( \frac{1}{b-a} \int_a^b f(x) \, dx \) by the Intermediate Value Theorem (Theorem 2.11). \(\square\)

Example 5.4.1

Example 5.4.1. Prove that if \( f \) is continuous on \([a, b]\), \( a \neq b \), and if

\[
\int_a^b f(x) \, dx = 0,
\]

then \( f(x) = 0 \) at least once in \([a, b]\).

Proof. Since \( f \) is continuous on \([a, b]\), then by The Mean Value Theorem for Definite Integrals (Theorem 5.3) we have

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

for some \( c \in [a, b] \). Since we are given that \( \int_a^b f(x) \, dx = 0 \), then for this value \( c \) we have

\[
f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{1}{b-a} (0) = 0,
\]

so that \( f(c) = 0 \), as claimed. \(\square\)

Theorem 5.4(a)

Theorem 5.4(a). The Fundamental Theorem of Calculus, Part 1. If \( f \) is continuous on \([a, b]\) then the function

\[
F(x) = \int_a^x f(t) \, dt
\]

has a derivative at every point \( x \) in \([a, b]\) and

\[
dF\over dx = \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x).
\]

Proof. Notice that

\[
F(x+h) - F(x) = \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt = \int_x^{x+h} f(t) \, dt.
\]

So

\[
\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt.
\]
Theorem 5.4(a) (continued)

Proof (continued). Since $f$ is continuous, The Mean Value Theorem for Definite Integrals (Theorem 5.3) implies that for some $c \in [x, x + h]$ we have

$$f(c) = \frac{1}{h} \int_{x}^{x+h} f(t) \, dt.$$ 

Since $c \in [x, x + h]$, then $\lim_{h \to 0} f(c) = f(x)$ (since $f$ is continuous at $x$). Therefore

$$\frac{dF}{dx} = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$ 

$$= \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(t) \, dt$$ 

$$= \lim_{h \to 0} f(c) = f(x)$$

Exercise 5.4.46

Exercise 5.4.46 Find $dy/dx$ when $y = \int_{1}^{x} \frac{1}{t} \, dt$ where $x > 0$.

Solution. Since $f(t) = 1/t$ is continuous on interval $[x, 1]$ when $0 < x < 1$ and is continuous on interval $[1, x]$ when $1 < x$, then by The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)), we have

$$\frac{d}{dx} \left[ \int_{1}^{x} \frac{1}{t} \, dt \right] = \frac{1}{x}$$

or $$\frac{dy}{dx} = \frac{1}{x}$$

Exercise 5.4.48

Exercise 5.4.48 Find $dy/dx$ when $y = x \int_{2}^{x^2} \sin(t^3) \, dt$.

Solution. First, we let $u = u(x) = x^2$ so that $y$ is in the form $y = x \int_{2}^{u} \sin(t^3) \, dt$. Then by the Derivative Product Rule (Theorem 3.3.G) and The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)),

$$\frac{d}{dx} [y] = \frac{d}{dx} \left[ x \int_{2}^{u} \sin(t^3) \, dt \right]$$

$$= [1] \left( \int_{2}^{u} \sin(t^3) \, dt \right) + (x) \frac{d}{dx} \left[ \int_{2}^{u} \sin(t^3) \, dt \right]$$

$$= [1] \left( \int_{2}^{u} \sin(t^3) \, dt \right) + (x) \frac{du}{dx} \left[ \int_{2}^{u} \sin(t^3) \, dt \right]$$

$$= [1] \left( \int_{2}^{u} \sin(t^3) \, dt \right) + (x) \left[ \sin((u)^3) \frac{du}{dx} \right]$$

$$= \left( \int_{2}^{u} \sin(t^3) \, dt \right) + (x)[\sin((x^2)^3)][2x]$$

$$= \int_{2}^{x^2} \sin(t^3) \, dt + 2x^2 \sin(x^6)$$

Exercise 5.4.48 (continued)

Solution (continued). First, we let $u = u(x) = x^2$ so that $y$ is in the form $y = x \int_{2}^{u} \sin(t^3) \, dt$. Then by the Derivative Product Rule (Theorem 3.3.G) and The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)),

$$\frac{d}{dx} [y] = \left[ \int_{2}^{u} \sin(t^3) \, dt \right] + (x) \frac{d}{dx} \left[ \int_{2}^{u} \sin(t^3) \, dt \right]$$

$$= \left( \int_{2}^{u} \sin(t^3) \, dt \right) + (x)[\sin((x^2)^3)](2x)$$

$$= \int_{2}^{x^2} \sin(t^3) \, dt + 2x^2 \sin(x^6)$$
Exercise 5.4.54

Find $\frac{dy}{dx}$ when $y = \int_{2x}^{1} \sqrt{t} \, dt$.

Solution. Then by the Derivative Product Rule (Theorem 3.3.G) and The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)),

\[
\frac{d}{dx} [y] = \frac{d}{dx} \left[ \int_{2x}^{1} \sqrt{t} \, dt \right] = \frac{d}{dx} \left[ - \int_{1}^{2x} \sqrt{t} \, dt \right] \text{ by Theorem 5.2(1)}
\]

\[
= -\frac{d}{dx} \left[ \int_{1}^{u} \sqrt{t} \, dt \right] \text{ where } u = 2x
\]

\[
= -\frac{d}{du} \left[ \int_{1}^{u} \sqrt{t} \, dt \right] \cdot \frac{du}{dx} \text{ by the Chain Rule, Theorem 3.2}
\]

\[
= -\frac{\sqrt{u}}{2} \cdot \frac{du}{dx} = -\frac{\sqrt{2x}}{2} \cdot \frac{1}{2} \ln 2 \cdot 2^x = -\frac{\ln 2}{2} 2^{4x/3}
\]

Theorem 5.4(b)

Theorem 5.4(b). The Fundamental Theorem of Calculus, Part 2. If $f$ is continuous at every point of $[a, b]$ and if $F$ is any antiderivative of $f$ on $[a, b]$, then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

Proof. We know from the first part of the Fundamental Theorem (Theorem 5.4(a)) that

\[
G(x) = \int_{a}^{x} f(t) \, dt
\]

defines an antiderivative of $f$. Therefore if $F$ is any antiderivative of $f$, then $F(x) = G(x) + k$ for some constant $k$.

Theorem 5.4(b) (continued)

Theorem 5.4(b). The Fundamental Theorem of Calculus, Part 2. If $f$ is continuous at every point of $[a, b]$ and if $F$ is any antiderivative of $f$ on $[a, b]$, then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

Proof (continued). Therefore

\[
F(b) - F(a) = [G(b) + k] - [G(a) + k] = G(b) - G(a)
\]

\[
= \int_{a}^{b} f(t) \, dt - \int_{a}^{a} f(t) \, dt = \int_{a}^{b} f(t) \, dt - 0
\]

\[
= \int_{a}^{b} f(t) \, dt,
\]

as claimed.

Exercise 5.4.6

Evaluate the integral $\int_{-2}^{2} (x^3 - 2x + 3) \, dx$.

Solution. By The Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)), we just need an antiderivative $F$ of the integrand $f(x) = x^3 - 2x + 3$. We can take $F(x) = x^4/4 - x^2 + 3x$. Then we have

\[
\int_{-2}^{2} (x^3 - 2x + 3) \, dx = \left( \frac{x^4}{4} - x^2 + 3x \right)_{-2}^{2}
\]

\[
= \left( \frac{(2)^4}{4} - (2)^2 + 3(2) \right) - \left( \frac{(-2)^4}{4} - (-2)^2 + 3(-2) \right)
\]

\[
= 4 - 4 + 6 - 4 + 4 + 6 = 12.
\]
Exercise 5.4.14

**Exercise 5.4.14.** Evaluate the integral \( \int_{-\pi/3}^{\pi/3} \sin^2 t \, dt \). HINT: By a half-angle formula, \( \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \).

**Solution.** By the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)), we just need an antiderivative \( F(t) = \sin^2 t \). Since \( \sin^2 t = \frac{1 - \cos 2t}{2} = \frac{1}{2} - \frac{1}{2} \cos 2t \), we can take

\[
F(t) = \frac{1}{2} \left( t - \frac{\sin 2t}{2} \right) \quad \text{(see Table 4.2 entry 3 in Section 4.8).}
\]

Then we have

\[
\int_{-\pi/3}^{\pi/3} \sin^2 t \, dt = \left. \frac{1}{2} \left( t - \frac{\sin 2t}{2} \right) \right|_{-\pi/3}^{\pi/3}
\]

Exercise 5.4.22

**Exercise 5.4.22.** Evaluate the integral \( \int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy \).

**Solution.** We apply the Fundamental Theorem of Calculus, Part 2 (Theorem 5.4(b)). We modify the integrand first so that find an antiderivative. We have

\[
\int_{-3}^{-1} \frac{y^5 - 2y}{y^3} \, dy = \int_{-3}^{-1} \frac{y^2 - 2y^{-2}}{y^3} \, dy = \left( \frac{y^3}{3} - 2(-y^{-1}) \right) \Big|_{-3}^{-1}
\]

\[
= \left( \frac{y^3}{3} + 2 \right) \Big|_{-3}^{-1} = \left( \frac{(-1)^3}{3} + 2 \right) - \left( \frac{(-3)^3}{3} + \frac{2}{(-3)} \right)
\]

\[
= \left( -\frac{1}{3} - 2 \right) - \left( -9 + \frac{2}{3} \right) = 7 + \frac{1}{3} = \frac{22}{3}.
\]

Exercise 5.4.64

**Exercise 5.4.64.** Find the area of the shaded region:

**Solution.** We know that a definite integral over \([a, b]\) of a nonnegative function \( f \) is (by definition) the area under \( y = f(x) \) from \( a \) to \( b \). Notice that the desired area (in blue) is the area in a rectangle of width \( 1 + \pi/4 \) and height 2 minus the area under \( y = \sec^2 t \) from \(-\pi/4\) to 0 (in yellow) and minus the area under \( y = 1 - t^2 \) from 0 to 1 (in orange):

That is, the desired area is

\[
(1 + \pi/4)(2) - \int_{-\pi/4}^{0} \sec^2 t \, dt - \int_{0}^{1} (1 - t^2) \, dt.
\]
Exercise 5.4.64 (continued)

Solution (continued). . . the desired area is

\[
(1 + \pi/4)(2) - \int_{-\pi/4}^{0} \sec^2 t \, dt - \int_{0}^{1} 1 - t^2 \, dt
\]

\[
= 2 + \pi/2 - \tan t|_{-\pi/4}^{0} - (t - t^3/3)|_{0}^{1}
\]

\[
= 2 + \pi/2 - (\tan(0) - \tan(-\pi/4)) - ((1) - (1)^3/3) - ((0) - (0)^3/3)
\]

\[
= 2 + \pi/2 - (1 - 2/3) = \frac{1}{3} + \pi/2.
\]

\[\square\]

Exercise 5.4.82

Exercise 5.4.82. Find the linearization of \( g(x) = 3 + \int_{1}^{x^2} \sec(t - 1) \, dt \) at \( x = -1 \).

Solution. Recall that the linearization of \( g \) at \( x = a \) is

\[ L(x) = g(a) + g'(a)(x - a). \]

We have

\[
g'(x) = \frac{d}{dx} \left[ 3 + \int_{1}^{x^2} \sec(t - 1) \, dt \right]
\]

\[
= \frac{d}{dx} \left[ 3 + \int_{1}^{u} \sec(t - 1) \, dt \right] \frac{du}{dx} \text{ by the Chain Rule, where } u = x^2
\]

\[
= 0 + \sec(u - 1) \frac{du}{dx} \text{ by The Fundamental Theorem of Calculus,}
\]

Part 1 (Theorem 5.4(a))

\[
= \sec(x^2 - 1)[2x] = 2x \sec(x^2 - 1).
\]

Exercise 5.4.82 (continued)

Exercise 5.4.82. Find the linearization of \( g(x) = 3 + \int_{1}^{x^2} \sec(t - 1) \, dt \) at \( x = -1 \).

Solution (continued). With \( g(x) = 3 + \int_{1}^{x^2} \sec(t - 1) \, dt \) and

\[
g'(x) = 2x \sec^2(x^2 - 1),
\]

we have

\[
g(a) = g(-1) = 3 + \int_{1}^{(-1)^2} \sec(t - 1) \, dt = 3
\]

\[
g'(-1) = g'(-1) = 2(-1) \sec((-1)^2 - 1) = -2 \sec(0) = -2(1) = -2.
\]

So the linearization of \( g \) at \( x = a = -1 \) is

\[
L(x) = g(-1) + g'(-1)(x - (-1))
\]

\[
= (3) + (-2)(x - (-1)) = 3 - 2x - 2 = \frac{1}{3} + \pi/2.
\]

\[\square\]

Exercise 5.4.72

Exercise 5.4.72. Find a function \( f \) satisfying the equation

\[ f(x) = e^2 + \int_{1}^{x} f(t) \, dt. \]

Solution. First, we differentiation with respect to \( x \) to get

\[
\frac{d}{dx} [f(x)] = \frac{d}{dx} \left[ e^2 + \int_{1}^{x} f(t) \, dt \right] = f(x)
\]

by The Fundamental Theorem of Calculus, Part 1 (Theorem 5.4(a)). So

\[
f'(x) = f(x).
\]

Some functions satisfying this condition are functions of the form \( ke^x \) where \( k \) is some constant. Notice also that

\[
f(1) = e^2 + \int_{1}^{1} f(t) \, dt = e^2 + 0 = e^2.
\]

Now \( (ke^x)|_{x=1} = ke^1 = ke \), so with \( k = e \) we have

\[
f(x) = ee^x = e^{x+1}.
\]
Exercise 5.4.72 (continued)

Exercise 5.4.72. Find a function \( f \) satisfying the equation
\[
f(x) = e^2 + \int_1^x f(t) \, dt.
\]

Solution (continued). With \( f(x) = e^{x+1} \), we have that both
\[
f(1) = e^{(1)+1} = e^2 \quad \text{and} \quad (\text{by the Fundamental Theorem of Calculus, Part 2 (Theorem 54.(b))})
\]
\[
e^2 + \int_1^x f(t) \, dt = e^2 + \int_1^x e^{t+1} \, dt = e^2 + e^{t+1} \bigg|_{t=1}^{t=x}
\]
\[
e^2 + (e^{(x)+1} - e^{(1)+1}) = e^2 + e^{x+1} - e^2 = e^{x+1} = f(x),
\]
as desired. So one such function is \( f(x) = e^{x+1} \). \( \square \)

Exercise 5.4.74 (continued)

Exercise 5.4.74. Show that if \( k \) is a positive constant, then the area between the \( x \)-axis and one arch of the curve \( y = \sin kx \) is \( 2/k \).

Solution (continued). . . . So the area is
\[
A = \int_0^{\pi/k} \sin kx \, dx \quad \text{(since \( \sin kx \geq 0 \) for \( x \in [0, \pi/k] \)).}
\]
Evaluating the integral using the Fundamental Theorem of Calculus, Part 2 (Theorem 54.(b)) we have
\[
A = \int_0^{\pi/k} \sin kx \, dx = \left. -\frac{\cos kx}{k} \right|_0^{\pi/k} = -\frac{\cos k(\pi/k)}{k} - \frac{\cos k(0)}{k}
\]
\[
= -\frac{\cos \pi}{k} + \frac{\cos 0}{k} = -\left(1 \right) + \frac{1}{k} = \frac{2}{k},
\]
as claimed (where the antiderivative of \( \sin kx \) is given by Table 4.2(2) in Section 4.8). \( \square \)

Example 5.4.8

Example 5.4.8. Find the area of the region between the \( x \)-axis and the graph of \( f(x) = x^3 - x^2 - 2x, -1 \leq x \leq 2 \).

Solution. We need the sign of \( f(x) = x^3 - x^2 - 2x \) so that we can separate the region bounded by the \( x \)-axis and the graph of \( y = f(x) \) into a part where the function is positive and a part where the function is negative. Notice that
\[
f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2)
\]
so that \( f(x) = 0 \) for \( x = -1, 0, \) and \( 2 \). Since \( f \) is continuous (it is a polynomial function), then we perform a sign test of \( f \) as we did when applying the First and Second Derivative Tests in Chapter 4.
Example 5.4.8 (continued 1)

Example 5.4.8. Find the area of the region between the x-axis and the graph of \( f(x) = x^3 - x^2 - 2x \), \(-1 \leq x \leq 2\).

Solution (continued). Consider:

<table>
<thead>
<tr>
<th>interval</th>
<th>((-\infty, -1))</th>
<th>(-1, 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>test value</td>
<td>(-2)</td>
<td>-1/2</td>
</tr>
</tbody>
</table>
| \(f(k)\)    | \((-2)^3 - (-2)^2 - 2(-2) = -8\) | \((-1/2)^3 - (-1/2)^2 - 2(-1/2) = 5/8\)
| \(f(x)\)    | \(-\)               | \(\)   |

So \(f(x) \geq 0\) for \(x \in [-1, 0] \cup [2, \infty)\), and \(f(x) \leq 0\) for \(x \in (-\infty, -1] \cup [0, 2]\). In particular, on \([-1, 0]\) we have \(f(x) \geq 0\) (and the area between \(f\) and the x-axis is given by the integral of \(f\) over \([-1, 0]\)), and on \([0, 2]\) we have \(f(x) \leq 0\) (and the negative of the area between \(f\) and the x-axis is given by the integral of \(f\) over \([0, 2]\)).

Example 5.4.8 (continued 2)

Solution (continued). So the desired area is

\[
A = \int_{-1}^{0} f(x) \, dx + \left( -\int_{0}^{2} f(x) \, dx \right)
\]

\[
= \int_{-1}^{0} x^3 - x^2 - 2x \, dx - \int_{0}^{2} x^3 - x^2 - 2x \, dx
\]

\[
= \left. \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \right|_{-1}^{0} - \left. \left( \frac{x^4}{4} - \frac{x^3}{3} - x^2 \right) \right|_{0}^{2}
\]

\[
= \left( \frac{0^4}{4} - \frac{0^3}{3} - (0)^2 \right) - \left( \frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2 \right)
\]

\[
- \left( \left( \frac{(2)^4}{4} - \frac{(2)^3}{3} - (2)^2 \right) - \left( \frac{(0)^4}{4} - \frac{(0)^3}{3} - (0)^2 \right) \right)
\]

\[
= ((0) - (1/4 + 1/3 - 1)) - ((4 - 8/3 - 4) - (0))
\]

\[
= 5/12 - (-8/3) = 5/12 + 8/3 = 37/12.
\]

Example 5.4.8 (continued 3)

Example 5.4.8. Find the area of the region between the x-axis and the graph of \( f(x) = x^3 - x^2 - 2x \), \(-1 \leq x \leq 2\).

Solution (continued). ...So the desired area is

\( A = 5/12 - (-8/3) = 5/12 + 8/3 = 37/12 \). The textbook gives the following graph to illustrate how the area is calculated: