Exercise 5.5.20

Exercise 5.5.20. Evaluate: \( \int 3y\sqrt{7-3y^2} \, dy \).

Solution. We let \( f(y) = 7 - 3y^2 \) so that \( f'(y) = -6y \). We then have
\[
\int 3y\sqrt{7-3y^2} \, dy = \frac{1}{2} \int -6y(7-3y^2)^{1/2} \, dy = \frac{-1}{2} \int (f(y))^{1/2} f'(y) \, dy
\]
\[
= \frac{-1}{2} \left( \frac{1}{3/2} (f(y))^{3/2} \right) + C = \frac{-1}{3} (7-3y^2)^{3/2} + C.
\]
Alternatively, let \( u = 7 - 3y^2 \) so that \( du = -6y \, dy \) or \( \frac{1}{2} \, du = 3y \, dy \). We then have
\[
\int 3y\sqrt{7-3y^2} \, dy = \int \sqrt{u} \frac{-1}{2} \, du = \frac{-1}{2} \int u^{1/2} \, du = \frac{-1}{2} \left( \frac{2}{3} u^{3/2} \right) + C
\]
\[
= \frac{-1}{3} (7-3y^2)^{3/2} + C. \quad \square
\]

Exercise 5.5.6

Exercise 5.5.6. Evaluate: \( \int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx \).

Solution. We let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = 1 + \sqrt{x} = 1 + x^{1/2} \) so that \( du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx \) or \( 2 \, du = \frac{1}{\sqrt{x}} dx \). Then
\[
\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx = \int (1 + \sqrt{x})^{1/3} \frac{1}{\sqrt{x}} \, dx = \int u^{1/3} \, du
\]
\[
= 2 \int u^{1/3} \, du = 2 \left( \frac{3}{4} u^{4/3} \right) + C = \frac{3}{2} (1 + \sqrt{x})^{4/3} + C. \quad \square
\]
Exercise 5.5.32

**Exercise 5.5.32.** Evaluate: \( \int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz. \)

**Solution.** We let \( u \) be some function of \( z \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \sec z \) so that \( du = \sec z \tan z \, dz \).

Then
\[
\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz = \int (\sec z)^{1/2} \sec z \tan z \, dz = \int (\sec z)^{-1/2} \sec z \tan z \, dz
\]
\[
= \int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{u} + C = \frac{2\sqrt{\sec z} + C}{2}.
\]

\[
\square
\]

Exercise 5.5.60

**Exercise 5.5.60.** Evaluate: \( \int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta. \)

**Solution.** We want to let \( u \) be some function of \( \theta \) where we see a multiple of \( u' \) as part of the integrand. There appears to be no obvious such choice for \( u \). Notice from Table 4.2 (or Table 4.2.A) from Section 4.8, the integral most closely resembles entry #12 from Table 4.2.A:
\[
\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + C.
\]
So we try \( u = e^\theta \) and \( du = e^\theta \, d\theta \). We then have \( \frac{du}{u} = d\theta \) or \( \frac{du}{u} = d\theta \).

Then
\[
\int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta = \int \frac{1}{\sqrt{(e^\theta)^2 - 1}} \, d\theta = \int \frac{1}{\sqrt{u^2 - 1}} \, du
\]
\[
= \int \frac{1}{u\sqrt{u^2 - 1}} \, du = \sec^{-1}(u) + C = \sec^{-1}(e^\theta) + C.
\]

\[
\square
\]

Exercise 5.5.56

**Exercise 5.5.56.** Evaluate: \( \int \frac{\ln \sqrt{t}}{t} \, dt. \)

**Solution.** First we rewrite the integral as
\[
\int \frac{\ln \sqrt{t}}{t} \, dt = \int \frac{\ln t^{1/2}}{t} \, dt = \frac{1}{2} \int \ln t \, dt. \]

We now let \( u \) be some function of \( t \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \ln t \) so that \( du = \frac{1}{t} \, dt \). Then
\[
\int \frac{\ln \sqrt{t}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt = \frac{1}{2} \int \ln \frac{1}{t} \, dt
\]
\[
= \frac{1}{2} \int u \, du = \frac{1}{2} \left( \frac{1}{2} u^2 \right) + C = \frac{1}{4} (\ln t)^2 + C.
\]

\[
\square
\]

Example 5.5.7(c)

**Example 5.5.7(c).** Evaluate: \( \int \tan x \, dx. \)

**Solution.** First we rewrite the integral as \( \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx. \) We now let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \cos x \) so that \( du = -\sin x \, dx \) or \( -du = \sin x \, dx \). Then
\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \sin x \, dx = \int \frac{1}{\cos x} \, (-du)
\]
\[
= -\ln |u| + C = -\ln |\cos x| + C = \ln |(\cos x)^{-1}| + C = \ln |\sec x| + C.
\]

\[
\square
\]
Example 5.5.8(b)

Evaluate $\int \sec x \, dx$.

**Solution.** This one requires a trick. We rewrite the integral as

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx.$$

Now we let $u = \sec x + \tan x$ so that $du = (\sec x \tan x + \sec^2 x) \, dx$. Then

$$\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sec x + \tan x| + C. \square$$

Exercise 5.5.68

**Exercise 5.5.68.** Evaluate: $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx$:
(a) by first letting $u = x - 1$, followed by $v = \sin u$, and then $w = 1 + v^2$,
(b) by first letting $u = \sin(x - 1)$ followed by $v = 1 + u^2$, and (c) by letting $u = 1 + \sin^2(x - 1)$.

**Solution.** (a) Following the instructions, we let $u = x - 1$ so that $du = dx$. Then $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx = \int \sqrt{1 + v^2} v \, dv$. Finally, we let $w = 1 + v^2$ so that $dw = 2v \, dv$ or $dv/2 = w/2 = \sqrt{w} \, dv$. Then

$$\int \sqrt{1 + v^2} v \, dv = \int \sqrt{w} \, dw/2 = \frac{1}{2} \int w^{1/2} \, dw = \frac{1}{2} \left( \frac{2}{3} w^{3/2} \right) + C$$

$$= \frac{1}{3} (1 + v^2)^{3/2} + C = \frac{1}{3} (1 + (1 + u^2)^{3/2} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C. \square$$

Exercise 5.5.68 (continued 1)

**Exercise 5.5.68.** Evaluate: $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx$:
(b) by first letting $u = \sin(x - 1)$ followed by $v = 1 + u^2$, and (c) by letting $u = 1 + \sin^2(x - 1)$.

**Solution.** (b) Following the instructions, we let $u = \sin(x - 1)$ so that $du = \cos(x - 1) \, dx$. Then

$$\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx = \int \sqrt{1 + u^2} u \, du.$$

Next, we let $v = 1 + u^2$ so that $dv = 2u \, du$ or $du/2 = v/2 = \sqrt{v} \, du$. Then

$$\int \sqrt{1 + u^2} u \, du = \int \sqrt{v} \, dv/2 = \frac{1}{2} \int v^{1/2} \, dv = \frac{1}{2} \left( \frac{2}{3} v^{3/2} \right) + C$$

$$= \frac{1}{3} (1 + u^2)^{2/3} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C. \square$$

Exercise 5.5.68 (continued 2)

**Exercise 5.5.68.** Evaluate: $\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx$:
(c) by letting $u = 1 + \sin^2(x - 1)$.

**Solution.** (c) Following the instructions, we let $u = 1 + \sin^2(x - 1)$ so that $du = 2 \sin(x - 1) \cos(x - 1) \, dx$ or $du/2 = \sin(x - 1) \cos(x - 1) \, dx$. Then

$$\int \sqrt{1 + \sin^2(x - 1)} \sin(x - 1) \cos(x - 1) \, dx = \int \sqrt{u} \, du/2 = \frac{1}{2} \int u^{1/2} \, du$$

$$= \frac{1}{3} \left( \frac{2}{3} u^{2/3} \right) + C = \frac{1}{3} (1 + \sin^2(x - 1))^{2/3} + C. \square$$
Exercise 5.5.70

Exercise 5.5.70. Solve the initial value problem: \( \frac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x \), \( y'(0) = 4 \), \( y(0) = -1 \).

Solution. First, \( \frac{dy}{dx} \in \int \frac{d^2y}{dx^2} \, dx = \int 4 \sec^2 2x \tan 2x \, dx \). With \( u = \sec 2x \) we have \( du = \sec 2x \tan 2x \, dx \) or \( du/2 = \sec 2x \tan 2x \, dx \). So

\[
\int 4 \sec^2 2x \tan 2x \, dx = 4 \int \sec 2x \sec 2x \tan 2x \, dx
\]

\[
= 4 \int u \, \frac{du}{2} = 2 \int d\, u = 2 \left( \frac{1}{2} u^2 \right) + C = u^2 + C = \sec^2 (2x) + C.
\]

So \( \frac{dy}{dx} = y' = \sec^2 (2x) + k_1 \) for some constant \( k_1 \). Since \( y'(0) = 4 \) then \( y'(0) = \sec^2 (2(0)) + k_1 = \sec^2 (0) + k_1 = 1 + k_1 = 4 \), or \( k_1 = 3 \). Hence \( dy/dx = \sec^2 (2x) + 3 \).

Solution (continued). Next, \( y \in \int \frac{dy}{dx} \, dx = \int \sec^2 (2x) + 3 \, dx \). With \( u = 2x \) we have \( du = 2 \, dx \) or \( du/2 = dx \). Then

\[
\int \sec^2 (2x) + 3 \, dx = \int (\sec^2 (u) + 3) \, \frac{du}{2} = \frac{1}{2} \int (\sec^2 (u) + 3) \, du
\]

\[
= \frac{1}{2} (\tan u + 3u) + C = \frac{1}{2} \tan u + \frac{3}{2} u + C
\]

\[
= \frac{1}{2} \tan (2x) + \frac{3}{2} (2x) + C = \frac{1}{2} \tan (2x) + 3x + C.
\]

So \( y = \frac{1}{2} \tan (2x) + 3x + k_2 \) for some constant \( k_2 \). Since \( y(0) = -1 \) then
\( y(0) = \frac{1}{2} \tan (2(0)) + 3(0) + k_2 = -1 \), or \( k_2 = -1 \). Hence

\[
y = \frac{1}{2} \tan (2x) + 3x - 1. \quad \square
\]