Chapter 5. Integrals

5.5. Indefinite Integrals and the Substitution Method—Examples and Proofs
Exercise 5.5.20. Evaluate: \[ \int 3y \sqrt{7 - 3y^2} \, dy. \]

**Solution.** We let \( f(y) = 7 - 3y^2 \) so that \( f'(y) = -6y \). We then have

\[
\int 3y \sqrt{7 - 3y^2} \, dy = \frac{1}{-2} \int -6y(7 - 3y^2)^{1/2} \, dy = \frac{-1}{2} \int (f(y))^{1/2} f'(y) \, dy
\]

\[
= \frac{-1}{2} \left( \frac{1}{3/2} (f(y))^{3/2} \right) + C = \frac{-1}{3} (7 - 3y^2)^{3/2} + C.
\]
Exercise 5.5.20

Exercise 5.5.20. Evaluate: $\int 3y \sqrt{7-3y^2} \, dy$.

Solution. We let $f(y) = 7-3y^2$ so that $f'(y) = -6y$. We then have

$$\int 3y \sqrt{7-3y^2} \, dy = \frac{1}{2} \int -6y (7-3y^2)^{1/2} \, dy = \frac{-1}{2} \int (f(y))^{1/2} f'(y) \, dy$$

$$= \frac{-1}{2} \left( \frac{1}{3/2} (f(y))^{3/2} \right) + C = \frac{-1}{3} (7-3y^2)^{3/2} + C.$$ 

Alternatively, let $u = 7-3y^2$ so that $du = -6y \, dy$ or $\frac{-1}{2} \, du = 3y \, dy$. We then have

$$\int 3y \sqrt{7-3y^2} \, dy = \int \sqrt{u} \frac{-1}{2} \, du = \frac{-1}{2} \int u^{1/2} \, du = \frac{-1}{2} \left( \frac{2}{3} u^{3/2} \right) + C$$

$$= \frac{-1}{3} (7-3y^2)^{3/2} + C.$$  ☐
Exercise 5.5.20. Evaluate: \[ \int 3y \sqrt{7 - 3y^2} \, dy. \]

Solution. We let \( f(y) = 7 - 3y^2 \) so that \( f'(y) = -6y \). We then have

\[
\int 3y \sqrt{7 - 3y^2} \, dy = \frac{1}{2} \int -6y(7 - 3y^2)^{1/2} \, dy = \frac{-1}{2} \int (f(y))^{1/2} f'(y) \, dy
\]

\[
= \frac{-1}{2} \left( \frac{1}{3/2} (f(y))^{3/2} \right) + C = \frac{-1}{3} (7 - 3y^2)^{3/2} + C.
\]

Alternatively, let \( u = 7 - 3y^2 \) so that \( du = -6y \, dy \) or \( \frac{-1}{2} \, du = 3y \, dy \). We then have

\[
\int 3y \sqrt{7 - 3y^2} \, dy = \int \sqrt{u} \frac{-1}{2} \, du = \frac{-1}{2} \int u^{1/2} \, du = \frac{-1}{2} \left( \frac{2}{3} u^{3/2} \right) + C
\]

\[
= \frac{-1}{3} (7 - 3y^2)^{3/2} + C.\]
Theorem 5.6. The Substitution Rule. If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du.
\]

Proof. By the Chain Rule (Theorem 3.2), \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \) for \( F \) an antiderivative of \( f \) because

\[
\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x).
\]

With \( u = u(x) = g(x) \), then

\[
\int f(g(x))g'(x) \, dx = \int \frac{d}{dx}[F(g(x))] \, dx = F(g(x)) + C \text{ by Theorem 4.8}
\]

\[= F(u) + C = \int F'(d) \, du \text{ by Theorem 4.8}
\]

\[= \int f(u) \, du, \text{ as claimed.} \]
Theorem 5.6

**Theorem 5.6. The Substitution Rule.** If \( u = g(x) \) is a differentiable function whose range is an interval \( I \) and \( f \) is continuous on \( I \), then

\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du.
\]

**Proof.** By the Chain Rule (Theorem 3.2), \( F(g(x)) \) is an antiderivative of \( f(g(x))g'(x) \) for \( F \) an antiderivative of \( f \) because

\[
\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x) = f(g(x))g'(x).
\]

With \( u = u(x) = g(x) \), then

\[
\int f(g(x))g'(x) \, dx = \int \frac{d}{dx}[F(g(x))] \, dx = F(g(x)) + C \quad \text{by Theorem 4.8}
\]

\[
= F(u) + C = \int F'(d) \, du \quad \text{by Theorem 4.8}
\]

\[
= \int f(u) \, du, \quad \text{as claimed.} \quad \Box
\]
Exercise 5.5.6. Evaluate: $\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx$.

Solution. We let $u$ be some function of $x$ where we see a multiple of $u'$ as part of the integrand. We choose $u = 1 + \sqrt{x} = 1 + x^{1/2}$ so that $du = \frac{1}{2} x^{-1/2} \, dx = \frac{1}{2\sqrt{x}} \, dx$ or $2 \, du = \frac{1}{\sqrt{x}} \, dx$. Then

$$\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx = \int u^{1/3} \, 2 \, du = 2 \left( \frac{3}{4} u^{4/3} \right) + C = \frac{3}{2} (1 + \sqrt{x})^{4/3} + C.$$
Exercise 5.5.6. Evaluate: \[ \int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx. \]

\textbf{Solution.} We let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = 1 + \sqrt{x} = 1 + x^{1/2} \) so that
\[ du = \frac{1}{2} x^{-1/2} \, dx = \frac{1}{2\sqrt{x}} \, dx \text{ or } 2 \, du = \frac{1}{\sqrt{x}} \, dx. \]

Then
\[ \int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx = \int (1 + \sqrt{x})^{1/3} \frac{1}{\sqrt{x}} \, dx = \int u^{1/3} 2 \, du \]
\[ = 2 \int u^{1/3} \, du = 2 \left( \frac{3}{4} u^{4/3} \right) + C = \frac{3}{2} (1 + \sqrt{x})^{4/3} + C. \]
Exercise 5.5.6. Evaluate: \[ \int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx. \]

Solution. We let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = 1 + \sqrt{x} = 1 + x^{1/2} \) so that \[ du = \frac{1}{2} x^{-1/2} \, dx = \frac{1}{2\sqrt{x}} \, dx \text{ or } 2 \, du = \frac{1}{\sqrt{x}} \, dx. \] Then

\[
\int \frac{(1 + \sqrt{x})^{1/3}}{\sqrt{x}} \, dx = \int (1 + \sqrt{x})^{1/3} \frac{1}{\sqrt{x}} \, dx = \int u^{1/3} 2 \, du
\]

\[= 2 \int u^{1/3} \, du = 2 \left( \frac{3}{4} u^{4/3} \right) + C = \frac{3}{2} (1 + \sqrt{x})^{4/3} + C. \] \( \square \)
Exercise 5.5.32. Evaluate: $\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz$.

Solution. We let $u$ be some function of $z$ where we see a multiple of $u'$ as part of the integrand. We choose $u = \sec z$ so that $du = \sec z \tan z \, dz$. Then

$$\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz = \int (\sec z)^{-1/2} \sec z \, dz = \int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{\sec z} + C.$$
Exercise 5.5.32. Evaluate: \[ \int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz. \]

Solution. We let \( u \) be some function of \( z \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \sec z \) so that \( du = \sec z \tan z \, dz \).

Then

\[
\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz = \int \frac{1}{(\sec z)^{1/2}} \sec z \tan z \, dz = \int (\sec z)^{-1/2} \sec z \tan z \, dz
\]

\[= \int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{u} + C = 2\sqrt{\sec z} + C. \] \(\square\)
Exercise 5.5.32. Evaluate: \( \int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz \).

Solution. We let \( u \) be some function of \( z \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \sec z \) so that \( du = \sec z \tan z \, dz \).

Then

\[
\int \frac{\sec z \tan z}{\sqrt{\sec z}} \, dz = \int \frac{1}{(\sec z)^{1/2}} \sec z \tan z \, dz = \int (\sec z)^{-1/2} \sec z \tan z \, dz
\]

\[
= \int u^{-1/2} \, du = 2u^{1/2} + C = 2\sqrt{u} + C = 2\sqrt{\sec z} + C. \quad \Box
\]
Exercise 5.5.56

Exercise 5.5.56. Evaluate: \( \int \frac{\ln \sqrt{t}}{t} \, dt \).

Solution. First we rewrite the integral as
\[
\int \frac{\ln \sqrt{t}}{t} \, dt = \int \frac{\ln t^{1/2}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt.
\]
We now let \( u \) be some function of \( t \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \ln t \) so that \( du = \frac{1}{t} \, dt \).
Exercise 5.5.56

Evaluate: \[ \int \frac{\ln \sqrt{t}}{t} \, dt. \]

Solution. First we rewrite the integral as
\[ \int \frac{\ln \sqrt{t}}{t} \, dt = \int \frac{\ln t^{1/2}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt. \]
We now let \( u \) be some function of \( t \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \ln t \) so that \( du = \frac{1}{t} \, dt \). Then
\[ \int \frac{\ln \sqrt{t}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt = \frac{1}{2} \int \ln \frac{1}{t} \, dt = \frac{1}{2} \int u \, du = \frac{1}{2} \left( \frac{1}{2} u^2 \right) + C = \frac{1}{4} (\ln t)^2 + C. \] \( \square \)
Exercise 5.5.56. Evaluate: \[ \int \frac{\ln \sqrt{t}}{t} \, dt. \]

Solution. First we rewrite the integral as
\[ \int \frac{\ln \sqrt{t}}{t} \, dt = \int \frac{\ln t^{1/2}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt. \]
We now let \( u \) be some function of \( t \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \ln t \) so that \( du = \frac{1}{t} \, dt \). Then
\[ \int \frac{\ln \sqrt{t}}{t} \, dt = \frac{1}{2} \int \frac{\ln t}{t} \, dt = \frac{1}{2} \int \ln t \, \frac{1}{t} \, dt \]
\[ = \frac{1}{2} \int u \, du = \frac{1}{2} \left( \frac{1}{2} u^2 \right) + C = \frac{1}{4} (\ln t)^2 + C. \]
Exercise 5.5.60

Exercise 5.5.60. Evaluate: \( \int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta \).

Solution. We want to let \( u \) be some function of \( \theta \) where we see a multiple of \( u' \) as part of the integrand. There appears to be no obvious such choice for \( u \). Notice from Table 4.2 (or Table 4.2.A) from Section 4.8, the integral most closely resembles entry #12 (from Table 4.2.A):
\[
\int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + C.
\]
So we try \( u = e^\theta \) and \( du = e^\theta \, d\theta \). We then have \( \frac{du}{e^\theta} = d\theta \) or \( \frac{du}{u} = d\theta \).
Exercise 5.5.60

Exercise 5.5.60. Evaluate: \[ \int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta. \]

Solution. We want to let \( u \) be some function of \( \theta \) where we see a multiple of \( u' \) as part of the integrand. There appears to be no obvious such choice for \( u \). Notice from Table 4.2 (or Table 4.2.A) from Section 4.8, the integral most closely resembles entry #12 (from Table 4.2.A): 
\[ \int \frac{1}{x \sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + C. \]
So we try \( u = e^\theta \) and \( du = e^\theta \, d\theta \). We then have \( \frac{du}{e^\theta} = d\theta \) or \( \frac{du}{u} = d\theta \). Then
\[
\int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta = \int \frac{1}{\sqrt{(e^\theta)^2 - 1}} \, d\theta = \int \frac{1}{\sqrt{u^2 - 1}} \, \frac{du}{u} \\
= \int \frac{1}{u \sqrt{u^2 - 1}} \, du = \sec^{-1}(u) + C = \sec^{-1}(e^\theta) + C. \]
Exercise 5.5.60

Evaluate: \[ \int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta. \]

Solution. We want to let \( u \) be some function of \( \theta \) where we see a multiple of \( u' \) as part of the integrand. There appears to be no obvious such choice for \( u \). Notice from Table 4.2 (or Table 4.2.A) from Section 4.8, the integral most closely resembles entry #12 (from Table 4.2.A):

\[
\int \frac{1}{x\sqrt{x^2 - 1}} \, dx = \sec^{-1}(x) + C.
\]

So we try \( u = e^\theta \) and \( du = e^\theta \, d\theta \). We then have \( \frac{du}{e^\theta} = d\theta \) or \( \frac{du}{u} = d\theta \). Then

\[
\int \frac{1}{\sqrt{e^{2\theta} - 1}} \, d\theta = \int \frac{1}{\sqrt{(e^\theta)^2 - 1}} \, d\theta = \int \frac{1}{\sqrt{u^2 - 1}} \, \frac{du}{u} = \int \frac{1}{u\sqrt{u^2 - 1}} \, du = \sec^{-1}(u) + C = \sec^{-1}(e^\theta) + C.
\]

\[ \square \]
Example 5.5.7(c). Evaluate: $\int \tan x \, dx$.

**Solution.** First we rewrite the integral as $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$. We now let $u$ be some function of $x$ where we see a multiple of $u'$ as part of the integrand. We choose $u = \cos x$ so that $du = -\sin x \, dx$ or $-du = \sin x \, dx$. Then

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{u} (-du) = -\ln |u| + C = -\ln |\cos x| + C = \ln |\sec x| + C.$$
Example 5.5.7(c). Evaluate: \( \int \tan x \, dx \).

Solution. First we rewrite the integral as \( \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \). We now let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \cos x \) so that \( du = -\sin x \, dx \) or \(-du = \sin x \, dx \). Then

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \sin x \, dx = \int \frac{1}{u} (-du) = -\ln |u| + C = -\ln |\cos x| + C = \ln |(\cos x)^{-1}| + C = \ln |\sec x| + C.
\]
Example 5.5.7(c). Evaluate: \[ \int \tan x \, dx. \]

Solution. First we rewrite the integral as \[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx. \] We now let \( u \) be some function of \( x \) where we see a multiple of \( u' \) as part of the integrand. We choose \( u = \cos x \) so that \( du = -\sin x \, dx \) or \(-du = \sin x \, dx\). Then

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{\cos x} \sin x \, dx = \int \frac{1}{u} (-du)
\]

\[
= - \ln |u| + C = - \ln |\cos x| + C = \ln |(\cos x)^{-1}| + C = \ln |\sec x| + C. \quad \square
\]
Example 5.5.8(b). Integral of $\sec x$.

Example 5.5.8(b). Evaluate: $\int \sec x \, dx$.

Solution. This one requires a trick. We rewrite the integral as

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx.$$ 

Now we let $u = \sec x + \tan x$ so that $du = (\sec x \tan x + \sec^2 x) \, dx$. 

□
Example 5.5.8(b). Evaluate: \( \int \sec x \, dx \).

Solution. This one requires a trick. We rewrite the integral as

\[
\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx.
\]

Now we let \( u = \sec x + \tan x \) so that \( du = (\sec x \tan x + \sec^2 x) \, dx \). Then

\[
\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx
\]

\[
= \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx
\]

\[
= \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sec x + \tan x| + C. \quad \Box
\]
Example 5.5.8(b). Evaluate: \( \int \sec x \, dx \).

Solution. This one requires a trick. We rewrite the integral as
\[
\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx.
\]
Now we let \( u = \sec x + \tan x \) so that \( du = (\sec x \tan x + \sec^2 x) \, dx \). Then
\[
\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx = \int \frac{1}{\sec x + \tan x} (\sec x \tan x + \sec^2 x) \, dx = \int \frac{1}{u} \, du = \ln |u| + C = \ln |\sec x + \tan x| + C.
\]
Exercise 5.5.68

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(a) by first letting \( u = x - 1 \), followed by \( v = \sin u \), and then \( w = 1 + v^2 \),
(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and (c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (a) Following the instructions, we let \( u = x - 1 \) so that \( du = dx \). Then
\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du.
\]
Exercise 5.5.68

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(a) by first letting \( u = x - 1 \), followed by \( v = \sin u \), and then \( w = 1 + v^2 \),
(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and
(c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (a) Following the instructions, we let \( u = x - 1 \) so that \( du = dx \). Then
\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du.
\]
Next, we let \( v = \sin u \) so that \( dv = \cos u \, du \). Then
\[
\int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du = \int \sqrt{1 + v^2} \, v \, dv.
\]
Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(a) by first letting \( u = x - 1 \), followed by \( v = \sin u \), and then \( w = 1 + v^2 \),
(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and
(c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (a) Following the instructions, we let \( u = x - 1 \) so that \( du = dx \). Then
\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du.
\]
Next, we let \( v = \sin u \) so that \( dv = \cos u \, du \). Then
\[
\int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du = \int \sqrt{1 + v^2} v \, dv.
\]
Finally, we let \( w = 1 + v^2 \) so that \( dw = 2v \, dv \) or \( dw/2 = v \, dv \). Then
\[
\int \sqrt{1 + v^2} v \, dv = \int \sqrt{w} \, dw/2 = \frac{1}{2} \int w^{1/2} \, dw = \frac{1}{2} \left( \frac{2}{3} w^{3/2} \right) + C
\]
\[
= \frac{1}{3} (1+v^2)^{3/2} + C = \frac{1}{3} (1+(\sin u)^2)^{3/2} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C \quad \square
\]
Exercise 5.5.68

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(a) by first letting \( u = x - 1 \), followed by \( v = \sin u \), and then \( w = 1 + v^2 \),

(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and

(c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (a) Following the instructions, we let \( u = x - 1 \) so that \( du = dx \). Then

\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du.
\]

Next, we let \( v = \sin u \) so that \( dv = \cos u \, du \). Then

\[
\int \sqrt{1 + \sin^2(u) \sin(u) \cos(u)} \, du = \int \sqrt{1 + v^2 \, v} \, dv.
\]

Finally, we let \( w = 1 + v^2 \) so that \( dw = 2v \, dv \) or \( dw/2 = v \, dv \). Then

\[
\int \sqrt{1 + v^2 \, v} \, dv = \int \sqrt{w \, dw/2} = \frac{1}{2} \int w^{1/2} \, dw = \frac{1}{2} \left( \frac{2}{3} w^{3/2} \right) + C
\]

\[
= \frac{1}{3} (1 + v^2)^{3/2} + C = \frac{1}{3} (1 + (\sin u)^2)^{3/2} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C \].

□
Exercise 5.5.68 (continued 1)

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and (c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (b) Following the instructions, we let \( u = \sin(x - 1) \) so that \( du = \cos(x - 1) \, dx \). Then
\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + u^2} \, u \, du.
\]
Next, we let \( v = 1 + u^2 \) so that \( dv = 2u \, du \) or \( dv/2 = u \, du \). Then
\[
\int \sqrt{1 + u^2} \, u \, du = \int \sqrt{v} \, dv/2 = \frac{1}{2} \int v^{1/2} \, dv = \frac{1}{2} \left( \frac{2}{3} v^{2/3} \right) + C.
\]
\[
= \frac{1}{3} (1 + u^2)^{2/3} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C. \quad \square
\]
Exercise 5.5.68 (continued 1)

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(b) by first letting \( u = \sin(x - 1) \) followed by \( v = 1 + u^2 \), and (c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (b) Following the instructions, we let \( u = \sin(x - 1) \) so that \( du = \cos(x - 1) \, dx \). Then

\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{1 + u^2} \, u \, du.
\]

Next, we let \( v = 1 + u^2 \) so that \( dv = 2u \, du \) or \( dv/2 = u \, du \). Then

\[
\int \sqrt{1 + u^2} \, u \, du = \int \sqrt{v} \, dv/2 = \frac{1}{2} \int v^{1/2} \, dv = \frac{1}{2} \left( \frac{2}{3} v^{2/3} \right) + C
\]

\[
= \frac{1}{3} (1 + u^2)^{2/3} + C = \frac{1}{3} (1 + \sin^2(x - 1))^{3/2} + C. \quad \square
\]
Exercise 5.5.68 (continued 2)

Exercise 5.5.68. Evaluate: \[ \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \]
(c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (c) Following the instructions, we let \( u = 1 + \sin^2(x - 1) \) so that \( du = 2 \sin(x - 1)[\cos(x - 1)] \, dx \) or \( du/2 = \sin(x - 1) \cos(x - 1) \, dx \). Then

\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int u^{1/2} \, du
\]

\[
= \frac{1}{2} \left( \frac{2}{3} u^{2/3} \right) + C = \frac{1}{3} (1 + \sin^2(x - 1))^{2/3} + C. \quad \square
\]
Exercise 5.5.68 (continued 2)

Exercise 5.5.68. Evaluate: \( \int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx \):

(c) by letting \( u = 1 + \sin^2(x - 1) \).

Solution. (c) Following the instructions, we let \( u = 1 + \sin^2(x - 1) \) so that \( du = 2 \sin(x - 1) \cos(x - 1) \, dx \) or \( du/2 = \sin(x - 1) \cos(x - 1) \, dx \). Then

\[
\int \sqrt{1 + \sin^2(x - 1) \sin(x - 1) \cos(x - 1)} \, dx = \int \sqrt{u} \frac{du}{2} = \frac{1}{2} \int u^{1/2} \, du
\]

\[
= \frac{1}{2} \left( \frac{2}{3} u^{2/3} \right) + C = \frac{1}{3} (1 + \sin^2(x - 1))^{2/3} + C. \quad \square
\]
Exercise 5.5.70

Exercise 5.5.70. Solve the initial value problem: \( \frac{d^2 y}{dx^2} = 4 \sec^2 2x \tan 2x, \)
y\'(0) = 4, y(0) = -1.

Solution. First, \( \frac{dy}{dx} \in \int \frac{d^2 y}{dx^2} \, dx = \int 4 \sec^2 2x \tan 2x \, dx. \) With \( u = \sec 2x \)
we have \( du = \sec 2x \tan 2x [2] \, dx \) or \( du/2 = \sec 2x \tan 2x \, dx. \) So

\[
\int 4 \sec^2 2x \tan 2x \, dx = 4 \int \sec 2x \sec 2x \tan 2x \, dx
\]

\[
= 4 \int u \frac{du}{2} = 2 \int d \, du = 2 \left( \frac{1}{2} u^2 \right) + C = u^2 + C = \sec^2(2x) + C.
\]
Exercise 5.5.70

Exercise 5.5.70. Solve the initial value problem: \( \frac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x, \)
\( y'(0) = 4, \ y(0) = -1. \)

Solution. First, \( \frac{dy}{dx} \in \int \frac{d^2y}{dx^2} \ dx = \int 4 \sec^2 2x \tan 2x \ dx. \) With \( u = \sec 2x \)
we have \( du = \sec 2x \tan 2x[2] \ dx \) or \( du/2 = \sec 2x \tan 2x \ dx. \) So

\[
\int 4 \sec^2 2x \tan 2x \ dx = 4 \int \sec 2x \sec 2x \tan 2x \ dx
\]

\[
= 4 \int u \frac{du}{2} = 2 \int d \ du = 2 \left( \frac{1}{2} u^2 \right) + C = u^2 + C = \sec^2(2x) + C.
\]

So \( \frac{dy}{dx} = y' = \sec^2(2x) + k_1 \) for some constant \( k_1. \) Since \( y'(0) = 4 \) then
\( y'(0) = \sec^2(2(0)) + k_1 = \sec^2(0) + k_1 = 1 + k_1 = 4, \) or \( k_1 = 3. \) Hence
\( dy/dx = \sec^2(2x) + 3. \)
Exercise 5.5.70.

Solve the initial value problem: \( \frac{d^2 y}{dx^2} = 4 \sec^2 2x \tan 2x, \) 
\( y'(0) = 4, \ y(0) = -1. \)

**Solution.** First, \( \frac{dy}{dx} \in \int \frac{d^2 y}{dx^2} \ dx = \int 4 \sec^2 2x \tan 2x \ dx. \) With \( u = \sec 2x \) we have \( du = \sec 2x \tan 2x [2] \ dx \) or \( du/2 = \sec 2x \tan 2x \ dx. \) So

\[
\int 4 \sec^2 2x \tan 2x \ dx = 4 \int \sec 2x \sec 2x \tan 2x \ dx \\
= 4 \int u \frac{du}{2} = 2 \int d \, du = 2 \left( \frac{1}{2} u^2 \right) + C = u^2 + C = \sec^2(2x) + C.
\]

So \( \frac{dy}{dx} = y' = \sec^2(2x) + k_1 \) for some constant \( k_1. \) Since \( y'(0) = 4 \) then 
\( y'(0) = \sec^2(2(0)) + k_1 = \sec^2(0) + k_1 = 1 + k_1 = 4, \) or \( k_1 = 3. \) Hence 
\( dy/dx = \sec^2(2x) + 3. \)
Solution (continued). Next, \( y \in \int \frac{dy}{dx} \, dx = \int \sec^2(2x) + 3 \, dx \). With \( u = 2x \) we have \( du = 2 \, dx \) or \( du/2 = dx \). Then

\[
\int \sec^2(2x) + 3 \, dx = \int (\sec^2(u) + 3) \frac{du}{2} = \frac{1}{2} \int (\sec^2(u) + 3) \, du
\]

\[
= \frac{1}{2} (\tan u + 3u) + C = \frac{1}{2} \tan u + \frac{3}{2} u + C
\]

\[
= \frac{1}{2} \tan(2x) + \frac{3}{2}(2x) + C = \frac{1}{2} \tan(2x) + 3x + C.
\]

So \( y = \frac{1}{2} \tan(2x) + 3x + k_2 \) for some constant \( k_2 \). Since \( y(0) = -1 \) then \( y(0) = \frac{1}{2} \tan(2(0)) + 3(0) + k_2 = -1 \), or \( k_2 = -1 \). Hence

\[
y = \frac{1}{2} \tan(2x) + 3x - 1. \]
Exercise 5.5.70 (continued)

Solution (continued). Next, \( y \in \int \frac{dy}{dx} \, dx = \int \sec^2(2x) + 3 \, dx \). With \( u = 2x \) we have \( du = 2 \, dx \) or \( du/2 = dx \). Then

\[
\int \sec^2(2x) + 3 \, dx = \int (\sec^2(u) + 3) \frac{du}{2} = \frac{1}{2} \int (\sec^2(u) + 3) \, du
\]

\[
= \frac{1}{2} (\tan u + 3u) + C = \frac{1}{2} \tan u + \frac{3}{2} u + C
\]

\[
= \frac{1}{2} \tan(2x) + \frac{3}{2} (2x) + C = \frac{1}{2} \tan(2x) + 3x + C.
\]

So \( y = \frac{1}{2} \tan(2x) + 3x + k_2 \) for some constant \( k_2 \). Since \( y(0) = -1 \) then

\[
y(0) = \frac{1}{2} \tan(2(0)) + 3(0) + k_2 = -1, \text{ or } k_2 = -1.
\]

Hence

\[
y = \frac{1}{2} \tan(2x) + 3x - 1. \quad \square
\]