Chapter 2. Limits and Continuity

2.4 One-Sided Limits

Note. We have already encountered functions that "try" to pass through one point from the left and a different point from the right. Consider the function $f(x) = x/|x|$ graphed in Figure 2.24 below. As x approaches 0 from the left (i.e., the negative side), we see that the graph of $y = f(x)$ "tries" to pass through the point $(0, -1)$. As x approaches 0 from the right (i.e., the positive side), we see that the graph of $y = f(x)$ "tries" to pass through the point $(0, 1)$. Inspired by Dr. Bob's Anthropomorphic Definition of Limit, we should be able to make the claim of some type of "one-sided limit."

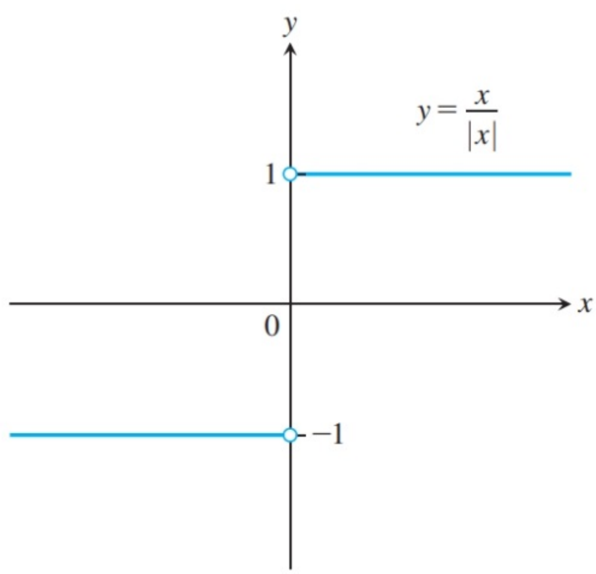


Figure 2.24

As with limits (or "two-sided limits") in Section 2.2, we have the following informal definition.

Definition. Informal Definition of Right-Hand and Left-Hand Limits.
Let $f(x)$ be defined on an interval (c,b), where $c<b$. If $f(x)$ approaches arbitrarily close to L as x approaches sufficiently close to c from within that interval, then we say that f has *right-hand limit* L at c, and write

$$\lim_{x\to c^+} f(x) = L.$$

Let $f(x)$ be defined on an interval (a,c), where $a<c$. If $f(x)$ approaches arbitrarily close to M as x approaches sufficiently close to c from within the interval (a,c), then we say that f has *left-hand limit* M at c, and we write

$$\lim_{x\to c^-} f(x) = M.$$

See Figure 2.25.

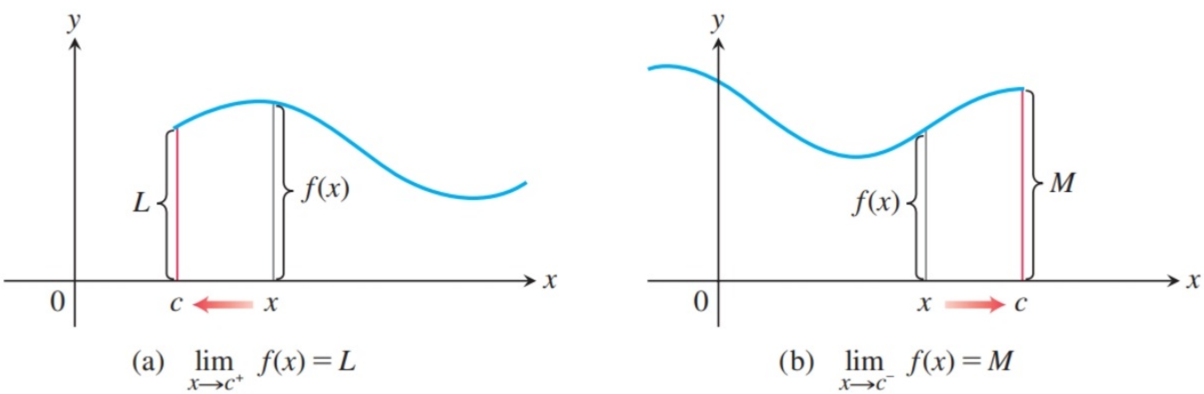


Figure 2.25

Examples. Example 2.4.1 and Exercise 2.4.10.

Note. Having described one-sided limits anthropomorphically and informally, we are now ready for a formal definition in terms of ε's and δ's.

Definition. Formal Definitions of One-Sided Limits.

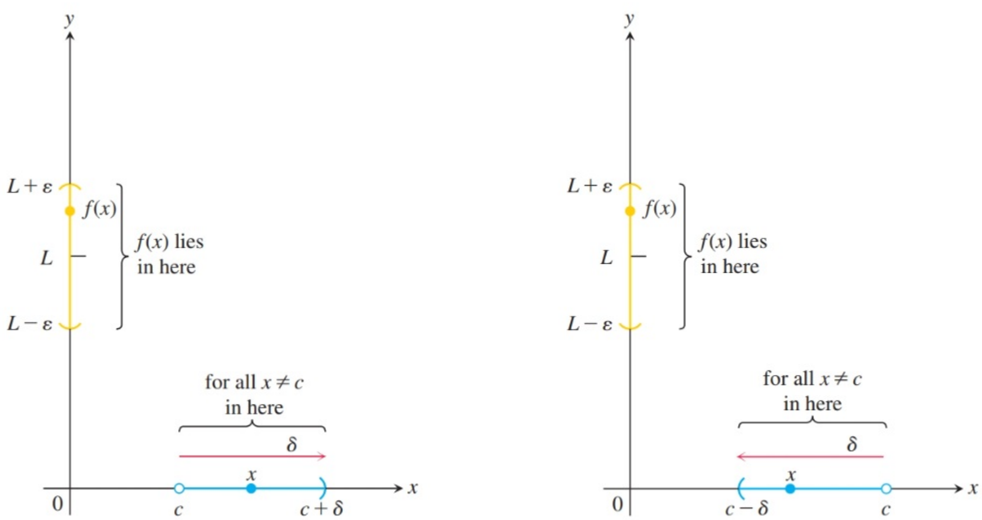
Let $f(x)$ be defined on an interval (c, b), where $c < b$. We say that $f(x)$ has *right-hand limit L at c*, and write $\lim_{x \to c^+} f(x) = L$ if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c < x < c + \delta \text{ implies } |f(x) - L| < \varepsilon.$$

Let $f(x)$ be defined on an interval (a, c), where $a < c$. We say that $f(x)$ has *left-hand limit L at c*, and write $\lim_{x \to c^-} f(x) = L$ if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$c - \delta < x < c \text{ implies } |f(x) - L| < \varepsilon.$$

Note. The text book illustrates the above formal definitions as given in Figures 2.28 and 2.29. We could also draw yellow and blue bands, as we did in the formal definition of two sided limits.



Figures 2.28 and 2.29

Note. We can also draw yellow and blue bands, as we did in the formal definition of two-sided limits. Similar to the discussion in Section 2.3, the graph of $y = f(x)$ must intersect a vertical side of the little green box (where the horizontal and vertical bands intersect) and does not intersect the horizontal sides of the little green box (except possibly at the corners). In the limit from the left the graph must intersect the left vertical edge of the little green box, and in the limit from the right the graph must intersect the right vertical edge of the little green box. Notice that neither of the δ's shown can be made any larger, since larger δ values will violate this intersection requirement between $y = f(x)$ and the little green boxes.

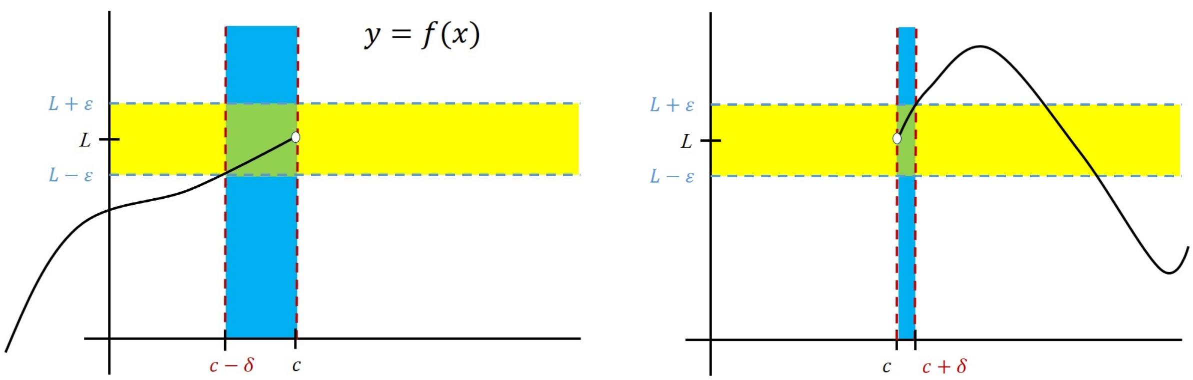


Figure. A limit as x approaches c from the left and right.

Example. Example 2.4.3. Prove that $\lim_{x\to 0^+} \sqrt{x} = 0$. (Notice that $\lim_{x\to 0} \sqrt{x}$ does not exist.)

Note. Each of the seven results of Theorem 2.1 for two-sided limits (the Sum Rule, the Product Rule, the Quotient Rule, the Power Rule, the Root Rule, etc.) also hold for one-sided limits. The theorems for limits of polynomials, rational functions, and the Sandwich Theorem (Theorems 2.2, 2.3, and 2.4) also hold for one-sided limits.

Note. Notice that if $\lim_{x\to c^+} f(x) = L$ then we require that f is defined on an interval (c, b) where $c < b$, and if $\lim_{x\to c^-} f(x) = L$ then we require that f is defined on an interval (a, c) where $a < c$. If both of these hold, then we have f defined on the open interval (a, b) except possibly at c itself. Notice that in none of these limits (one-sided and two-sided) does it "matter" what happens *at* $x = c$! But if both of the one-sided limits (for $x \to c^-$ and $x \to c^+$) exist and are the same, then the two-sided limit (for $x \to c$) exists, and conversely. This is given in the next theorem.

Theorem 2.6. Relation Between One-Sided and Two-Sided Limits. Suppose that a function f is defined on an open interval containing c, except possibly at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \text{ if and only if both } \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

Example. Exercise 2.4.50.

Example. We now look at Example 2.4.4, but modify it slightly. We consider the function $f(x) = \sin(\pi/x)$. A graph looks like the graph given in Figure 2.31.

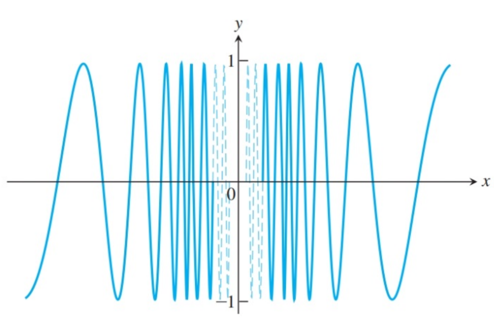


Figure 2.31

We can see that there is no particular point that the graph of $y = f(x)$ "tries" to pass through for x close to 0. So by Dr. Bob's Anthropomorphic Definition of Limit, we claim that $\lim_{x\to 0} f(x)$ does not exist, and similarly (with a one-sided anthropomorphic idea) neither $\lim_{x\to 0^-} f(x)$ nor $\lim_{x\to 0^+} f(x)$ exist. This is correct, but not what happens when we make a table of function values for x near 0:

x	$f(x)$	x	$f(x)$	x	$f(x)$
1	0	1/5	0	1/9	0
1/2	0	1/6	0	1/10	0
1/3	0	1/7	0	1/100	0
1/4	0	1/8	0	1/1000	0

So the table might make us think that the function has a limit as $x \to 0^+$ of 0. But as we see from the graph, this is not the case! This is why making tables of function values can be misleading!!! When considering a limit, the behavior of a function at a particular set of x values is not what matters; it matters what the function does *for all* x values "close to" c (namely, the x values satisfying $0 < |x - c| < \delta$ for a two-sided limit or the x values satisfying $c - \delta < x < c$ or $c < x < c + \delta$ for one-sided limits).

Note. We now use The Sandwich Theorem (Theorem 2.4) to address a limit involving trigonometric functions.

Theorem 2.7. Limit of the Ratio $(\sin\theta)/\theta$ as $\theta \to 0$.

For θ in radians,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

Example. Example 2.4.5(a): Show that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

Examples. Exercises 2.4.28 and 2.4.52.

Revised: 7/18/2020