

Chapter 11. Infinite Sequences and Series

11.1 Sequences

Note. We now shift gears and consider a more theoretical, less tangible concept than those with which we have recently dealt.

Definition. An infinite *sequence* of numbers is a function whose domain is the set of positive integers. We denote the sequence $\{f(n) \mid n \in \mathbb{N}\}$ as $\{a_n\}$ where $f(n) = a_n$.

Definition. The sequence $\{a_n\}$ *converges* to the number L if for every $\epsilon > 0$ there exists an integer N such that for all $n > N$ we have

$$|a_n - L| < \epsilon.$$

If no such number L exists, then the sequence $\{a_n\}$ *diverges*. If $\{a_n\}$ converges to L , we write $\lim_{n \rightarrow \infty} a_n = L$ and call L the *limit* of the sequence.

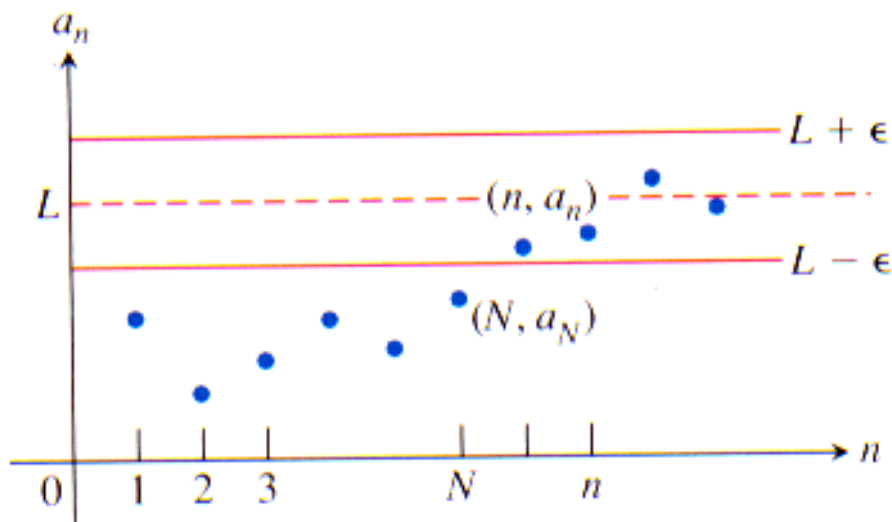


Figure 11.2 page 749

Example 1a. Let $a_n = \frac{1}{n}$. Prove that $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. Let $\epsilon > 0$ be given. Let N be an integer greater than $1/\epsilon$. Then for all $n > N$ we have $0 < a_n = 1/n < 1/N < \epsilon$, or $|a_n - 0| < \epsilon$.

Therefore $\lim_{n \rightarrow \infty} a_n = 0$. *Q.E.D.*

Definition. The sequence $\{a_n\}$ *diverges to infinity* if for every number M there is an integer N such that for all n larger than N , $a_n > M$. If

this condition holds, we write

$$\lim_{n \rightarrow \infty} a_n = \infty \text{ or } a_n \rightarrow \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ *diverges to negative infinity* and write

$$\lim_{n \rightarrow \infty} a_n = -\infty \text{ or } a_n \rightarrow -\infty.$$

Example. Page 757 Number 30.

Theorem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers and let A and B be real numbers. If $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$ then

1. *Sum Rule:* $\lim_{n \rightarrow \infty} (a_n + b_n) = A + B.$

2. *Difference Rule:* $\lim_{n \rightarrow \infty} (a_n - b_n) = A - B.$

3. *Product Rule:* $\lim_{n \rightarrow \infty} (a_n b_n) = AB.$

4. *Constant Multiple Rule:* $\lim_{n \rightarrow \infty} (k b_n) = k B.$

5. *Quotient Rule:* $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B},$ if $B \neq 0.$

Note. The proofs for each of these is similar to the proofs of the corresponding results for functions.

Example. Page 757 Number 28.

Theorem 2. The Sandwich Theorem for Sequences.

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N and is $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$ also.

Example. Page 757 Number 42.

Theorem 3. Let $\{a_n\}$ be a sequence of real numbers. If $a_n \rightarrow L$ and if f is a function that is continuous at L and defined at all a_n , then $f(a_n) \rightarrow f(L)$.

Example. Page 752 Example 5.

Theorem 4. Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$.

Then $\lim_{x \rightarrow \infty} f(x) = L$ implies that $\lim_{n \rightarrow \infty} a_n = L$.

Note. Theorem 4 allows us to use L'Hôpital's Rule on sequences.

Example. Page 757 Number 50.

Theorem 5. The following six sequences converge to the limits listed below:

1. $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0.$

2. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1.$

3. $\lim_{n \rightarrow \infty} x^{1/n} = 1$ for $x > 0.$

4. $\lim_{n \rightarrow \infty} x^n = 0$ for $|x| < 1.$

5. $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

6. $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$

Definition. A sequence can be defined recursively by giving:

1. The value(s) of the initial term or terms and
2. A rule, called a *recursion formula*, for calculating any later term from terms that precede it.

Example 11b. The *Fibonacci sequence* is defined recursively as: $a_1 = a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$. The first few terms therefore are: 1, 1, 2, 3, 5, 8, 13, 21, 44, 65, ...

Definition. A sequence $\{a_n\}$ with the property that $a_n \leq a_{n+1}$ for all n is called a *nondecreasing sequence*.

Definition. A sequence $\{a_n\}$ is *bounded from above* if there exists a number M such that $a_n \leq M$ for all n . The number M is an *upper bound* for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the *least upper bound* for $\{a_n\}$.

Theorem 6. A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Note. The proof of Theorem 6 depends heavily on the very definition of the real numbers. Appendix A.4 gives some details in the form of the *Completeness Property*. This property states that every *set* of real numbers with an upper bound, has a least upper bound.

Example. Page 759 Number 100.