Chapter 11. Infinite Sequences and Series

11.10 Applications of Power Series

Note. If we define $f(x) = (1+x)^m$, then we find that the Taylor series for f is

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define (for **any** m)

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}.$$

This is called the *binomial series.* It converges for $|x| < 1$.

Example. Page 831 Number 10.

Note. When we cannot find a relatively simple expression for the solution of an initial value problem or differential equation, we try to get information about the solution in other ways. One way is to try to find a power series representation for the solution. If we do so, we immediately have a source of polynomial approximations of the solution, which may be all we really need.

Example. Page 832 Number 16. Notice that the solution is $y = e^{2x}$.

Example. We can use series to evaluate definite and indefinite integrals. For example, consider Page 832 Number 43.

Note. We can take the binomial series and replace x with x^2 to find that

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Integrating we find that for $|x| \leq 1$ (see page 828 for details)

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Plugging $x = 1$ into this formula, we find that we have a series representation for π: $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Notice that this is an alternating series, and hence we could approximate π by taking partial sums of this series and applying the Alternating Series Estimation Theorem. When you hear that people have calculated π to a million digits, or some such, then they are using a method similar to this approach.

Note. We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.

Example. Page 829 Example 8. Evaluate $\lim_{x \to 0} \dfrac{\sin x - \tan x}{x^3}$ using

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \tan x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)}.$$