Chapter 11. Infinite Sequences and Series

11.11 Fourier Series

Note. When investigating the problem of heat conduction in a long thin insulated rod, French mathematician Jean-Baptiste Joseph Fourier needed to express a function $f(x)$ as a trigonometric series. Generally, if $f(x)$ is defined on the interval $[0, 2\pi]$, we need to know the coefficients a_0, a_k and b_k $(k \geq 1)$ for which

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) .$$

This equation is called a *Fourier series* for f on the interval $(0, 2\pi)$.

Note. We can verify (see Exercises 9 through 13) that for positive integers p and q:

1. $$\int_0^{2\pi} \cos px \cos qx \, dx = \begin{cases} 0, & p \neq q, \\ \pi, & p = q \end{cases} ,$$

2. $$\int_0^{2\pi} \cos px \sin qx \, dx = 0,$$

3. $$\int_0^{2\pi} \sin px \sin qx \, dx = \begin{cases} 0, & p \neq q, \\ \pi, & p = q \end{cases}$$

Note. Calculation of a_0. We integrate the Fourier Series of f from 0 to 2π and assume that the operations for integration and summation can be interchanged to give:

$$\int_0^{2\pi} f(x)\,dx = a_0 \int_0^{2\pi} dx + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos kx\,dx + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin kx\,dx.$$

For every positive integer k, the last two integrals on the right-hand side are zero. Therefore

$$\int_0^{2\pi} f(x)\,dx = a_0 \int_0^{2\pi} dx = 2\pi a_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx.$$

Note. Calculation of a_m. We multiply both sides of the Fourier series representation of f by $\cos mx$, $m > 0$, and integrate the result from 0 to 2π:

$$\begin{aligned} \int_0^{2\pi} f(x)\cos mx\,dx &= a_0 \int_0^{2\pi} \cos mx\,dx \\ &\quad + \sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos kx \cos mx\,dx \\ &\quad + \sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin kx \cos mx\,dx. \end{aligned}$$

Applying formulas 1, 2, and 3 we reduce this equation to

$$\int_0^{2\pi} f(x)\cos mx\,dx = a_m \int_0^{2\pi} \cos mx \cos mx\,dx = \pi a_m.$$

Therefore

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx.$$

Note. Calculation of b_m. We multiply both sides of the Fourier series representation of f by $\sin mx$, $m > 0$, and integrate the result from 0 to 2π:

$$\begin{aligned}\int_0^{2\pi} f(x)\sin mx\,dx \;&=\; a_0\int_0^{2\pi} \sin mx\,dx \\ &\quad+\sum_{k=1}^{\infty} a_k \int_0^{2\pi} \cos kx \sin mx\,dx \\ &\quad+\sum_{k=1}^{\infty} b_k \int_0^{2\pi} \sin kx \sin mx\,dx.\end{aligned}$$

Applying formulas 1, 2, and 3 we reduce this equation to

$$\int_0^{2\pi} f(x)\sin mx\,dx = b_m \int_0^{2\pi} \sin mx \sin mx\,dx = \pi b_m.$$

Therefore

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx.$$

Definition. The constants a_0, a_k, b_k are called the *Fourier coefficients* of f.

Example. Example 1 page 835. Find the Fourier series for

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \pi \\ 2, & \pi < x \leq 2\pi \end{cases}$$

Solution. The graph of f is:

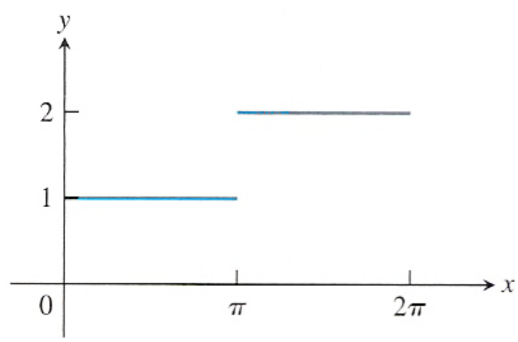

Figure 11.16a page 836.

Computing the Fourier coefficients, we find that (see pages 836 and 837 for computations):

$$f(x) = \frac{3}{2} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}$$

The graph of the first few partial sums of the Fourier series are:

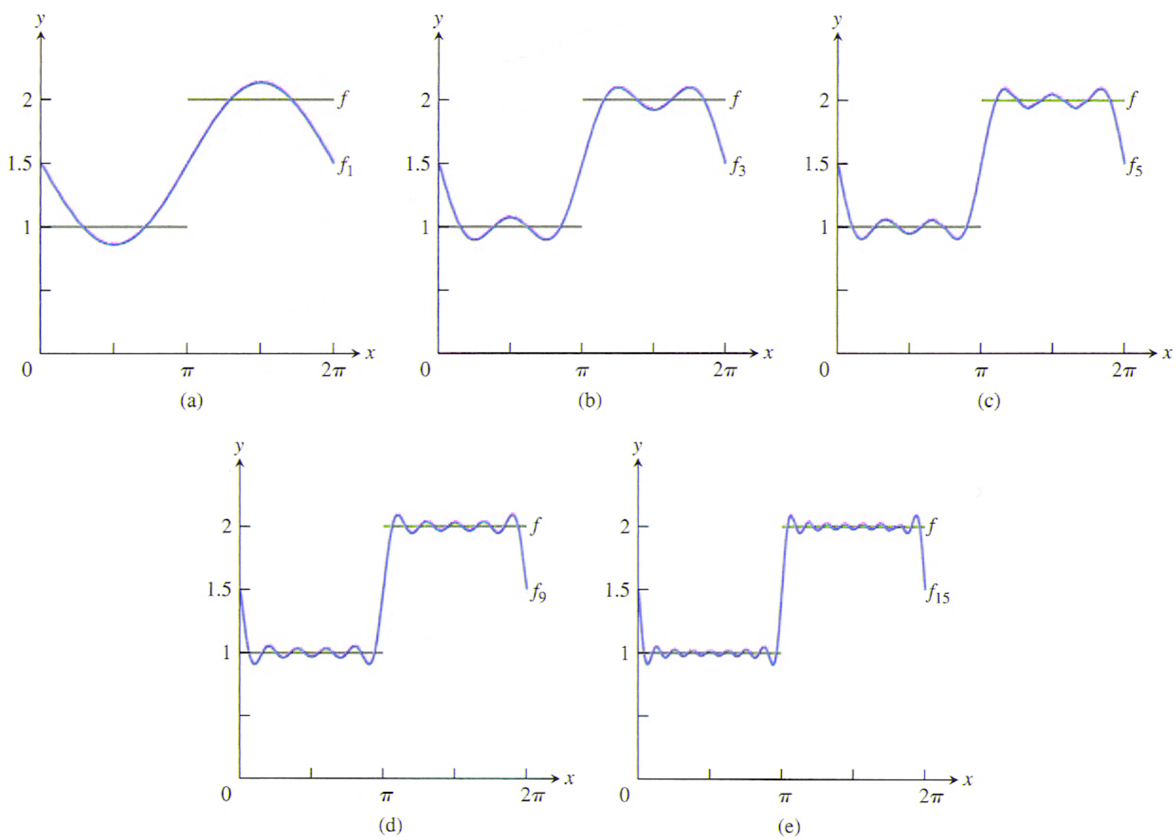

Figure 11.17 page 837.

Note. We notice from the above figure that each of the partial sums of the Fourier series passes through the point $x = \pi$, $y = 3/2$. The reason for this is given in the following theorem.

Theorem 24. If the function f and its derivative f' are piecewise continuous over the interval $-L < x < L$, then f equals its Fourier series at all points of continuity. At a point c where a jump discontinuity occurs in f, the Fourier series converges to the average

$$\frac{\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)}{2} \equiv \frac{f(c^-) + f(c^+)}{2}.$$