Chapter 11. Infinite Sequences and Series

11.8 Taylor and Maclaurin Series

Note. We now want to see that if a function has derivatives of all orders (it is said to be *infinitely differentiable*), then can we construct a power series for it? If we assume that a function has a power series representation

$$f(x) = \sum_{n=1}^{\infty} a_n (x-a)^n$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence, we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots \\ f''(x) &= 1 \cdot 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_4(x-a)^2 + \cdots \\ &\quad + n(n-1)a_n(x-a)^{n-1} + \cdots \\ f'''(x) &= 1 \cdot 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4(x-a) + 3 \cdot 4 \cdot 5a_5(x-a)^5 + \cdots \\ &\quad + n(n-1)(n-2)a_n(x-a)^{n-2} + \cdots \end{aligned}$$

with the n^{th} derivative for all n being

$$f^{(n)}(x) = n!a_n + \text{ a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations hold at $x = a$, we have

$$\begin{aligned} f'(a) &= a_1 \\ f''(a) &= 1 \cdot 2a_2 \\ f'''(a) &= 1 \cdot 2 \cdot 3a_3 \\ &\vdots \\ f^{(n)}(x) &= n!a_n. \end{aligned}$$

Therefore if f has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ then we have $a_n = \dfrac{f^{(n)}(a)}{n!}$. So we must have

$$f(x) = f(x) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

Definition. Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the *Taylor series generated by f at $x = a$* is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The *Maclaurin series generated by f* is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Example. Page 810 Number 14.

Definition. Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the *Taylor polynomial of order* n generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$
$$+ \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note. Just as the linearization of f at $x = a$ provides the best approximation of f in the neighborhood of a, the higher-order Taylor polynomials provide the best polynomial approximation of their respective degrees. For example, consider the following graph of $y = e^x$ along with the Taylor polynomials P_1, P_2, and P_3 centered at $a = 0$.

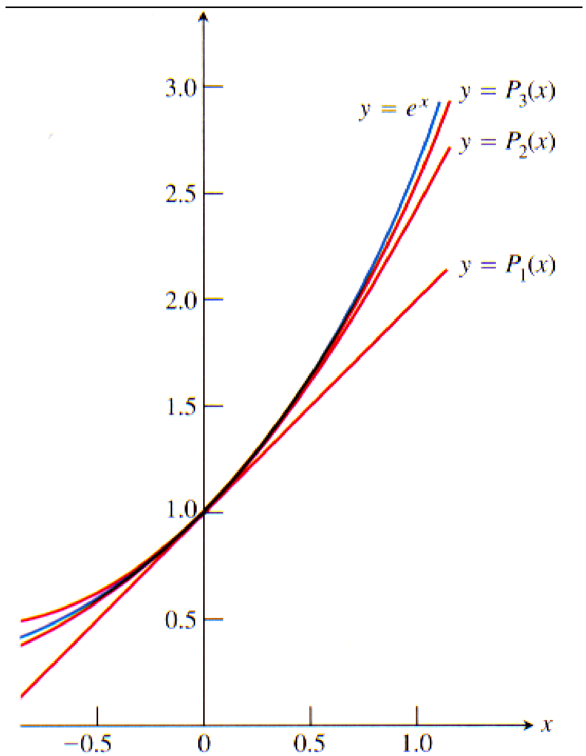

Figure 11.12 page 808

Example. Page 80 Example 3. Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$. The Taylor polynomials have the following graphs:

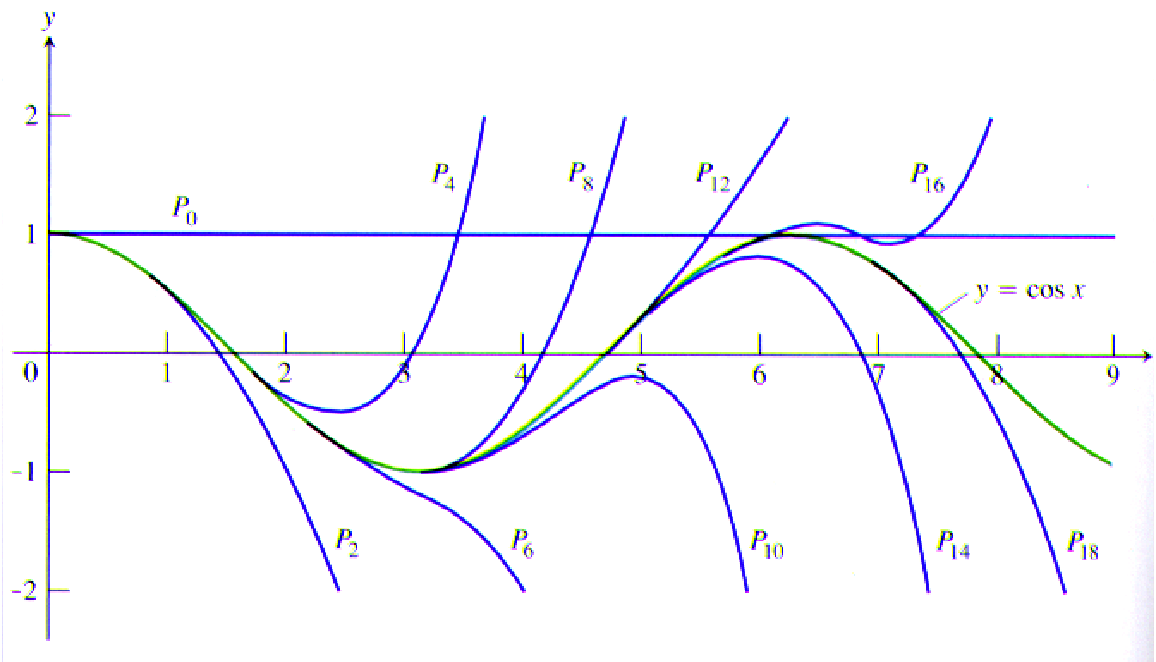

Figure 11.13 page 809

Example. Page 810 Example 4. Consider

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

Then we can show (by definition) that the derivatives of f of all orders when evaluated at $x = 0$ are 0: $f^{(n)}(0) = 0$ for all nonnegative integers n. Therefore the Maclaurin series for f is

$$f(0) + f'(0)x + \frac{f''(0)}{2!} + \cdots + \frac{f^{(n)}(0)}{n!} + \cdots = 0.$$

There is a problem here since this series equals f for $x \leq 0$, but does not equal f for $x > 0$. Therefore the condition of having a power series representation is stronger than the condition of being infinitely differentiable. This is a classical example of a function which is infinitely differentiable (on all of $\mathbb{R}$), but has no series representation (valid on all of $\mathbb{R}$).

Example. Page 811 Number 31.