Chapter 11. Infinite Sequences and Series

11.9 Convergence of Taylor Series; Error Estimates

Note. We still have unanswered questions relevant to the generation of Taylor series from infinitely differentiable functions:

1. When does a Taylor series converge to its generating function?

2. How accurately do a function's Taylor polynomials approximate the funtion on a given interval?

Theorem 22. Taylor's Theorem

If f and its first n derivatives f', f'', $\ldots$, $f^n$are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c betwee a and b such that

$$f(b) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}+\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Theorem. Taylor's Formula

If f is differentiable through order $n+1$ in an open interval I containing a, then for each $x \in I$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!} + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \text{ for some } c \text{ between } a \text{ and } x.$$

Note. If we can be insured that the remainder term R_n goes to 0 as $n \to \infty$, then the Taylor series will converge to the generating function. This is summarized in the following theorem.

Theorem 23. The Remainder Estimation Theorem.

If there are positive constants M and r such that $|f^{(n+1)}(t)| \leq Mr^{n+1}$ for all t between a and x, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!.}$$

If these conditions hold for every n and all the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Example. Page 820 number 20.

Note. We can establish the following Maclaurin series:

1. $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots x^n = \displaystyle\sum_{n=0}^{\infty} x^n \quad (|x| < 1)$

2. $\dfrac{1}{1+x} = 1 - x + x^2 + \cdots + (-x)^n + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n \quad (|x| < 1)$

3. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$

4. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots + (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$ (all x)

5. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^n}{(2n)!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$ (all x)

6. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots - (-1)^{n-1} \dfrac{x^n}{n} + \cdots \displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{x^n}{n} \quad (-1 < x \leq 1)$

7. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots + (-1)^n \dfrac{x^{2n+1}}{2n+1} \quad (|x| \leq 1)$

Example. Page 819 number 8; page 820 number 32.

Example. Find a series for $f(x) = e^{-x^2}$. Use the series to approximate $\displaystyle\int_0^1 e^{-x^2}\, dx$ to the nearest 0.01.